DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - STATISTICS
FALL 2010

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each question is worth 25 points.

1. Consider the pdf, $f(x;\gamma) = \exp\{-(x-\gamma)\}\, 1_{x>\gamma}$, and suppose $x_1,\ldots,x_n$ is an iid sample from that distribution. You may use the following fact without proof. If $f(x)$ is a pdf and $F(x)$ is a cdf, then the kth order statistic from a sample of size n has pdf $\frac{n!}{(k-1)!(n-k)!} f(x)F(x)^{k-1}(1-F(x))^{n-k}$.

   (a) Find a sufficient statistic for γ in this distribution.

   (b) Find a maximum likelihood estimator for γ.

   (c) Modify the MLE to get an unbiased estimator of γ.

   (d) Suggest a method of moments estimator for γ.

2. Suppose that $x_1,\ldots,x_n$ is an iid sample from a Normal(μ,1) distribution. As a reminder, the normal density is $f(x|\mu,1) = \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(x-\mu)^2}{2}\right\}$.

   (a) Prove that
   $$\exp\left\{-\sum_{i=1}^{n}(x_i-\mu)^2\right\} = \exp\left\{-(n-1)s^2 + n(\overline{x}-\mu)^2\right\}$$
   where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2$.

   (b) What is the likelihood of μ?

   (c) Let μ have a $Normal(\theta,\tau^2)$ prior ($f(\mu) = \frac{1}{\tau\sqrt{2\pi}}\exp\left\{\frac{-(\mu-\theta)^2}{2\tau^2}\right\}$). Derive the posterior density of μ, i.e. $f(\mu|x_1,\ldots,x_n)$.

   (d) What is the mean of the posterior distribution of μ?

   (e) What is a 95% interval for μ?

3. Suppose you are interested in the assessing whether or not flipping a coin is ”fair.” (i.e. the probability of heads equals the probability of tails). Suppose you flip the coin n times. Let X_n be the number of heads, and let $\widehat{P} = X_n/n$ be an estimator of the probability of heads.

   (a) Prove or disprove that your estimator is unbiased.

   (b) Prove or disprove that your estimator is consistent.

   (c) Suppose $n = 1000$ and $X_{1000} = 400$. What is an approximate 95% interval for the probability of heads?

   (d) Based on your answer to (c), do you think the coin is fair? Why or why not?

4. A liter of pond water is treated with x units of a chemical to purify it. (x is on the log scale, so x could be negative.) After treatment, the water is either clean ($Y = 1$) or not ($Y = 0$). Suppose the water needs to be treated by at least Z units of the chemical to clean it, but Z is unknown and depends on the particular sample of water. (Z is on the same scale as x.) Further, assume that Z has a standard normal distribution. Note that Z is random (the randomness comes from the particular sample of water) and x is fixed.

   (a) What is the probability that the water is clean after treatment?

   (b) What is the distribution of Z given that the water is not clean after treatment?

   (c) Derive the moment generating function for Z given that $Y = 0$. You may express your answer as an unsimplified integral that involves the standard normal pdf and other functions.

   (d) Use the result from the previous part to derive an expression for $E(Z|Y = 0)$.