DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
FALL 2010

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each question is worth 20 points.

1. Suppose you are told to toss a die until you have observed each of the six faces.

   (a) Let Y_1 be the trial on which the first face is tossed, Y_2 be the number of additional tosses required to get a face different than the first, Y_3 be the number of additional tosses required to get a face different than the first two distinct faces, ..., and Y_6 be the number of additional tosses required to get the last remaining face after all other faces have been observed. Find the distribution of each $Y_i, i = 1, \cdots, 6$.

   (b) What is the expected number of tosses required in order to observe each of the six faces?

2. (a) Suppose $X \sim N(0, 1)$. Find the p.d.f. of $Y = X^2$.

   (b) Let X_1 and X_2 be two independent random variables; X_1 has an exponential distribution with mean 1, and X_2 has an exponential distribution with mean 2. Find the p.d.f. of $Y = 2X_1 + X_2$.

   (c) Let X_1 and X_2 be two independent exponentially distributed random variables, each with mean 1. Find $P(X_1 > X_2 | X_1 < 2X_2)$.

3. Let Z be a standard normal random variable and let $Y_1 = Z$ and $Y_2 = Z^2$.

   (a) Find $E(Y_1)$, $E(Y_2)$, and $E(Y_1 Y_2)$.

   (b) Find $Cov(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

4. Suppose that $X_1, \cdots, X_k$ are iid $N(\mu, \sigma^2)$, $k \geq 2$. Denote:

$$U_1 = \sum_{i=1}^{k} X_i, U_j = X_1 - X_j \text{ for } j = 2, \cdots, k$$

   (a) Show that $\mathbf{U} = (U_1, \cdots, U_k)$ has a k-dimensional normal distribution;

   (b) Show that U_1 and $(U_2, \cdots, U_k)$ are independent;

   (c) Express S^2 as a function of $U_2, \cdots, U_k$ alone. Hence, show that $\bar{X}$ and S^2 are independently distributed. (*Hint: You may use the fact that* $\binom{k}{2} S^2 = \sum_{1 \leq i < j \leq k} \frac{1}{2}(X_i - X_j)^2$).

5. (a) Let $\{\xi_n, n \geq 1\}$ be a sequence of independently identically distributed random variables with $E(\xi_1) = \mu$, $Var(\xi_1) = \sigma^2 < \infty$, and $P(\xi_1 = 0) = 0$. Prove that

$$\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2} \to \frac{\mu}{\mu^2 + \sigma^2}, \qquad n \to \infty,$$

in probability. (*Hint: you may use the theorem that says if X_n converges to X in probability and Y_n converges to Y in probability, and if f is continuous, then $f(X_n, Y_n)$ converges to $f(X, Y)$ in probability. If further $X = a$ and $Y = b$ are constants, then f only needs to be continuous at (a, b).*).

(b) Let

$$X_n = \begin{cases} n & \text{with probability } 1/n \\ 0 & \text{with probability } 1 - 1/n. \end{cases}$$

Show that X_n converges in probability to zero, but $E(X_n)$ and $Var(X_n)$ do not converge to zero.