Department of Mathematics and Statistics, UMass Amherst
Basic Numeric Analysis Exam, September 2010

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correcct
PhD: 75% with at least three substantially correct.

1. Consider the one-point method

$$x_{n+1} = g(x_n),\ n = 0, 1, 2, \ldots \qquad \text{with} \qquad g(x) = a_1 x + a_2 x^2.$$

   (a) For a positive number a, find the constants a_1, a_2, such that the above method converges to $\sqrt{a}$ quadratically, given that the initial guess is close enough to $\sqrt{a}$.
   (b) Find the maximal possible interval for the initial guess x_0, which guarantees that the iteration converge to $\sqrt{a}$.

2. Consider the following quadrature rule:

$$I(f) = \omega_1 f(x_1) + \omega_2 f(-\frac{1}{2}) \approx \int_{-1}^{1} f(x)dx, \qquad x_1 \in [-1, 1].$$

   Find ω_1, ω_2, and x_1, such that the quadrature is exact for polynomials of degree less than or equal to n with the largest n. Justify your answer.

3. Let $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ for $x \in \mathbb{R}^n$. For any matrix $A = \{a_{ij}\}_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$, define the norm of A by

$$\|A\|_1 = \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1}.$$

   Prove that

$$\|A\|_1 = \max_{j} \sum_{i=1}^{n} |a_{ij}| \quad \text{(maximum absolute column sum)}.$$

4. For the step function

$$f(x) = \begin{cases} 1, & x \in [-1, 0], \\ -1, & x \in (0, 1], \end{cases}$$

   solve the following minimization problem

$$\min_{a_0,a_1,\ldots,a_n} \int_{-1}^{1} \left(f(x) - \sum_{k=0}^{n} a_k P_k(x) \right)^2 dx,$$

   and find the minimum. Here $\{a_k\}$ are constants, and $\{P_k(x)\}$ are the Legendre polynomials which are orthogonal on $[-1, 1]$:

$$\int_{-1}^{1} P_k(x) P_l(x) dx = \frac{2}{2k+1}\delta_{kl}.$$

   The Legendre polynomials can be obtained by the following recurrence relation:

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad (m+1)P_{m+1}(x) - (2m+1)xP_m(x) + mP_{m-1}(x) = 0.$$

5. Consider the well-posed initial value problem for $0 \le t \le 1$

$$\begin{cases} y' = f(t, y)\,, \\ y(0) = y_0\,. \end{cases}$$

where f is a smooth function. Let $0 = t_0 < t_1 < \ldots < t_N = 1$ be a uniform partition of the interval $[0, 1]$ with step size $h = 1/N$. For a constant parameter $\theta \in [0, 1]$, introduce the generalized mid-point method

$$y_{n+1} = y_n + h[(1-\theta) f(t_n, y_n) + \theta f(t_{n+1}, y_{n+1})]\,.$$

(a) Determine the order of the method.

(b) Show that the method is absolutely stable when $\theta \in [1/2, 1]$.

6. Define function $f(x)$ as

$$f(x) = \begin{cases} \sin x, & x \in [0, x_0], \\ \cos x, & x \in (x_0, 2\pi]. \end{cases}$$

Show that the maximum error for polynomial interpolation of any degree will be NOT less than $|\sin x_0 - \cos x_0|/2$.

7. Solve the equation $\frac{dy}{dt} = y + t$ with a simple splitting as

$$\begin{aligned} \frac{dy}{dt} &= t, \\ \frac{dy}{dt} &= y. \end{aligned}$$

Both equations after splitting can be solved exactly. This gives us a numerical scheme as follows

$$\begin{cases} y^* - y_n &= \frac{1}{2} t_{n+1}^2 - \frac{1}{2} t_n^2, \\ y_{n+1} &= e^h y^*, \end{cases} \qquad n = 0, 1, 2, \ldots$$

where $h = (x_{n+1} - x_n)$ is the time-step size. Find the leading term of the local truncation error, and determine the order of the scheme.