Department of Mathematics and Statistics
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Basic Exam - Complex Analysis

August 2010

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Calculate, for real β,
$$\oint_C \frac{e^{\beta z}}{z^2\,(z^2+2z+2)}$$
around the circle $C = \{\, z : |z| = 3 \,\}$ in the counterclockwise direction.

2. (a) How many zeros, counted with multiplicities, does $f(z) = z^7 - 5z^3 + 12$ have in the annulus $A = \{\, z \in \mathbb{C} : 1 < |z| < 2 \,\}$?

   (b) Let
$$f(z) = \frac{(z-2)^2\,(z^2+3z+6-2i)}{z^3}.$$
   Evaluate
$$\frac{1}{2\pi i}\oint_{|z|=3} \frac{f'(z)}{f(z)}\,dz$$
   where the integral is taken in the counterclockwise direction.

3. (a) Let $f(z) = \tan(z)$. Show that f maps the the vertical strip
$$S = \{\, w \in \mathbb{C} : -\pi/4 < \mathrm{Re}(w) < \pi/4 \,\}$$
   one-to-one onto the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\,\}$.

   (b) Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is entire with $\mathrm{Re}(f)$ bounded above. Prove that f must be constant.

4. Let U be the portion of the open upper half-plane $H = \{\, z \in \mathbb{C} : \mathrm{Im}(z) > 0 \,\}$ that lies outside the unit circle. Construct a one-to-one conformal function f that maps U onto H. Give an explicit, simple formula for $f(z)$.

5. Find a Laurent series expansion of
$$f(z) = \frac{1}{z^2 - 5z + 6}$$
in powers of z that is valid on some region containing the point $1+2i$, and determine the largest region over which the series you found converges to the function f.

6. Let B be the open unit ball, H the open upper half-plane, and I the interval $\{x \in \mathbb{R} \ : \ -1 < x < 1\}$. Suppose $f : (B \cap H) \cup I \to \mathbb{C}$ is continuous on its domain and holomorphic on $B \cap H$. Suppose, further, that f is real-valued on I. Prove that f has a holomorphic extension to B.

   Note: The above statement is a special case of the *Reflection Principle*, and you are asked to prove it **without** using the Reflection Principle. Do not forget to prove that f is holomorphic at points of the interval I.

   Hint: Among the many proofs, a particularly elementary one uses a version of Morera's Theorem involving contour integrals along rectangles with sides parallel to the axis.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function that has periods 1 and τ with $\mathrm{Im}(\tau) > 0$. Thus $f(z + j + k\tau) = f(z)$ for all $z \in C$ and all $j, k \in \mathbb{Z}$.

   (a) Prove that if f has no singularities whatsoever, then f must be constant.

   (b) Assume that f has no poles on the boundary C of the set

   $$S = \{\, s + t\tau \ : 0 \le s < 1,\ 0 \le t < 1\}.$$

   Prove that $\displaystyle\int_C \frac{f'(z)}{f(z)}\, dz = 0.$

   (c) Assume now that f is not constant. Prove that the total number of poles of f in S, counting multiplicities, is at least 2; in other words, prove that either f has a pole of order at least 2 or else f has at least two poles.

8. Let a and b be positive real numbers. Evaluate the integral

   $$\int_0^\infty \frac{\cos(bx)}{x^2 + a^2} dx$$

   using residues. Include a sketch of the contours you use and proofs of all the estimates involved.

9. (a) Prove that the series

   $$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

   defines a meromorphic function $f(z)$, periodic with period 1, over the complex plane.

   (b) Prove that $g(z) := f(z) - \dfrac{\pi^2}{\sin^2(\pi z)}$ is an entire function.

10. Let D denote the open set $D := \{z \ : \ |z| > 1\}$ and $\overline{D} := \{z \ : \ |z| \geq 1\}$ its closure. Suppose $f(z)$ is holomorphic on an open set U containing $\overline{D}$ and that

$$\lim_{z \to \infty} f(z) \ = \ 1.$$

Show that for any z in D,

$$\int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta \quad = \quad 2\pi i(1 - f(z)).$$