BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Denote by X the set of six vectors

$$(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1),\ (0,1,1,0),\ (0,1,0,1),\ (0,0,1,1).$$

Find two different, non-empty subsets Y_1, Y_2 of X such that

- the elements of each Y_i are linearly independent, and
- the elements of $Y_i \cup \{\vec{x}\}$ are not linearly indepedent for any $\vec{x} \in X \setminus Y_i$.

Justify your answer!

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2. Let $\vec{w} \in \mathbf{R}^n$ be a unit vector. Define a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ as follows:

$$T\vec{x} := \vec{x} - 2(\vec{x} \cdot \vec{w})\vec{w}$$

(where $\vec{x} \cdot \vec{w}$ is the usual inner product in $\mathbf{R}^n$).

(a) Show that T is an *orthogonal* transformation, in other words $||T\vec{x}|| = ||\vec{x}||$ for all $\vec{x}$.

Hint: What is the geometric interpretation of T? You might want to draw a picture.

(b) Find the Jordan form of A.

3(a) Let A, B be $n \times n$ matrices. If $AB = 0$, show that

$$\text{rank}(A) + \text{rank}(B) \leq n.$$

(b) For any $n \times n$ matrix A, show that there exists a $n \times n$ real matrix B with

$$AB = 0 \quad \text{and} \quad \text{rank}(A) + \text{rank}(B) = n.$$

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4. Suppose A is a real $n \times n$ matrix with all entries ≥ 0 and with the sum of entries in each column equal to 1.

(a) Show that A has an eigenvector with eigenvalue equal to 1.

(b) Show that all eigenvalues λ of A satisfy $|\lambda| \leq 1$

Hint: One way to do this is prove the corresponding statement for A^t; of course there are other ways.

Part II. Advanced Calculus

1. The *Fundamental Theorem of Arithmetic* says that every integer $n > 1$ can be written uniquely as
$$n = p_1^{e_1} \cdots p_r^{e_r},$$
where $p_1 < \cdots < p_r$ are primes and the e_i are positive integers. Use the Fundamental Theorem to show that if $\{n_i\}_{i \in \mathbf{N}}$ is an infinite, strictly increasing sequence of positive integers such that the series $\sum_{i=1}^{\infty} 1/n_i$ diverges, then the set
$$\{p \text{ prime} : \ p \text{ divides } n_i \text{ for some } i\}$$
is infinite.

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2. Fix numbers $R > r > 0$. Compute the volume of the solid obtained by rotating the circle $(x - R)^2 + y^2 = r^2$ above the y-axis. Show your work.

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3. Let $f_1(x, y), f_2(x, y)$ be smooth functions on $\mathbf{R}^2$. Denote by X_i the surface in $\mathbf{R}^3$ defined by $z = f_i(x, y)$. Suppose $X_1 \cap X_2 = \emptyset$. As p_i runes through all points on X_i, show that the line segment $\overline{p_1 p_2}$ is perpendicular to both X_i whenever the *length* of the line segment reaches a local minimum or local maximum.

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4. Let $f : [0, 1] \to \mathbf{R}$ be a Riemann integrable function. It is a fact that for any integer $n > 0$, the function $g_n(x) := f(x^n)$ is also Riemann integrable on $[0, 1]$.

(a) If f is continuous at $x = 0$, show that

$$\lim_{n \to \infty} \int_0^1 g_n(t)dt = f(0). \tag{1}$$

(b) Give an example to show that (1) is false if f is not continuous at $x = 0$.

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5. Let $f(x, y) = xy + \displaystyle\int_0^y \sin(t^2)dt$.

(a) Compute $\nabla f(a, b)$.

(b) Show that $(0, 0)$ is a saddle point of $f(x, y)$.
