University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
August 2010

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Let M be an n-dimensional manifold and $N \subset M$ an embedded submanifold of codimension k.

- **a)** Prove that the tangent bundle TN may be regarded as an embedded submanifold of TM of codimension $2k$.
- **b)** Find equations that describe TS^{n-1}, implicitly, as an embedded submanifold of $T\mathbb{R}^n \cong \mathbb{R}^{2n}$. That is, find a smooth map $F\colon \mathbb{R}^{2n} \to \mathbb{R}^2$, having $0 \in \mathbb{R}^2$ as a regular value, and such that
$$TS^{n-1} \ = F^{-1}(0).$$
Verify that the map you defined satisfies the required conditions.

Problem 2. Let M be a three-dimensional manifold. A *contact structure* on M is a two-dimensional C^∞ distribution Δ on M which, locally, is given as $\Delta = \ker(\alpha)$, where α is a one-form such that $\alpha \wedge d\alpha \neq 0$ everywhere in the open set where α is defined. That is, for every $p \in M$ there exists an open set U containing p and a one-form α defined on U such that
$$\Delta(q) = \ker(\alpha(q)),$$
for all $q \in U$ and $\alpha \wedge d\alpha \neq 0$ for all $q \in U$.

- **a)** Prove that if M has a contact structure, then M is orientable.
- **b)** Prove that a contact structure Δ is not involutive.
- **c)** Give an example of a contact structure on $\mathbb{R}^3$.

Problem 3. Prove or disprove the following statements:

- **a)** If M is a compact, orientable, n-dimensional manifold (without boundary) then $H^n_{dR}(M) \neq 0$.
- **b)** If M is a compact, 2-dimensional manifold (without boundary) then there exists a C^∞ map $F\colon \mathbb{R}^3 \to \mathbb{R}$ such that 0 is a regular value of F and $M \cong F^{-1}(0)$.

Problem 4. Let M be a smooth manifold and $X \in \mathcal{X}(M)$, a C^∞ vector field on M.

- **a)** Prove that if M is compact then X is complete.
- **b)** Find the one-parameter group of diffeomorphisms (flow) of S^2 defined by the restriction to S^2 of the vector field in $\mathbb{R}^3$:
$$X \ = \ z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\,.$$

Problem 5. Let M be an open set of $\mathbb{R}^2$ with the Riemannian metric whose matrix in the standard frame ∂_x, ∂_y is given by:

$$\begin{pmatrix} \lambda(x,y) & 0 \\ 0 & \lambda(x,y) \end{pmatrix},$$

where λ is a positive, smooth function on M.

a) Compute $\operatorname{grad}(f)$, for $f \in C^\infty(M)$ in terms of the standard frame ∂_x, ∂_y.
b) Compute $\operatorname{div}(X)$, where $X \in \mathcal{X}(M)$ is a C^∞ vector field on M.
c) Compute the Laplacian $\Delta(f)$, where $f \in C^\infty(M)$.

Problem 6. We identify $\mathbb{R}^3$ with the three-dimensional Heisenberg group:

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \; : \; x, y, z \in \mathbb{R} \right\} \subset GL(3, \mathbb{R}).$$

a) Find a basis (frame) of left-invariant vector fields on $G \cong \mathbb{R}^3$.
b) Find a nowhere-zero, left-invariant form of degree 3 on $G \cong \mathbb{R}^3$.

Problem 7. Let $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half-plane with the Poincaré metric:

$$g = y^{-2}(dx \otimes dx + dy \otimes dy).$$

a) Find the geodesic through the point $p = (0, 1)$ whose tangent vector at p is the vector $(0, a)$.
b) Prove that for every $q \in H$ and every $v \in T_q(H)$ there exists a (globally defined) geodesic

$$\gamma \colon \mathbb{R} \to H$$

such that $\gamma(0) = q$ and $\gamma'(0) = v$.

Problem 8. Let M be the surface in $\mathbb{R}^2$ defined by:

$$M \;=\; \{(u \cos v, u \sin v, v) : u, v \in \mathbb{R}\} \subset \mathbb{R}^3$$

with the metric induced by $\mathbb{R}^3$.

a) Compute the Gaussian curvature of M.
b) Given that the Christoffel symbols of M relative to the coordinates (u, v) are all zero except:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{u}{1+u^2} \; ; \quad \Gamma^1_{22} = -u\,,$$

find the geodesic, parametrized by arc-length, joining the points $(0, 0, 0)$ and $(0, 0, 1)$ in M.