Department of Mathematics and Statistics
University of Massachusetts Amherst

Advanced Exam – Algebra
Fall 2010

Passing Standard: It is sufficient to do five problems correctly, including at least one from each of the three parts.

1. Group Theory and Representation Theory

1. Let G be a finite group and let V be an irreducible complex representation of G.

(a) Let $x \in V$, $x \neq 0$. Prove that $\dim V \leq [G : G_x]$. (Here G_x is the stabilizer of x for the action of G on V.)

(b) Let $H \subset G$ be an Abelian subgroup. Prove that

$$\dim V \leq [G : H].$$

2. Let p be a prime number and let G be a finite p-group. Let $H \subset G$ be a proper subgroup. Prove that the normalizer of H in G is larger than H:

$$N_G(H) \neq H.$$

3. Let X be a set with at least two points. Let G be a group acting doubly transitively on a set X: that is, for any $x_1, x_2 \in X$ and $y_1, y_2 \in X$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$, there is a $g \in G$ such that $gx_1 = y_1$ and $gx_2 = y_2$. Show that for any $x \in X$, the stabilizer G_x is a maximal proper subgroup of G. (That is, $G_x \neq G$ and there are no proper subgroups H of G such that $G_x \subsetneq H$.)

2. Commutative Algebra

4. Let R be a commutative ring and let M be an R-module. Recall that M is called *flat* if, for any short exact sequence of R-modules

$$0 \to N' \to N \to N'' \to 0,$$

the induced sequence

$$0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$

is also exact.

(a) Let M be a flat R-module, let $r \in R$ be a non-zero-divisor, and let $m \in M$ be such that $rm = 0$. Prove that $m = 0$.

(b) Prove that an R-module M is flat if and only if the localization $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for any prime ideal $\mathfrak{p} \subset R$.

5. Let R be a commutative domain. Show that if $R[x]$ is a principal ideal domain, then R is a field.

6. Let R denote a commutative ring containing a field F. Suppose that R is finite dimensional as an F-vector space.

(a) Prove that any prime ideal of R is maximal.

(b) Prove that R has finitely many maximal ideals.

3. Galois Theory

7. Let K be a field and let G be a finite group of automorphisms of K. Let $H \subset G$ be a subgroup. Prove that there exists $x \in K$ such that

$$H = \{g \in G \mid g \cdot x = x\}.$$

8. Let p be a prime number and let n be a positive integer. Prove that $\mathrm{GL}_n(\mathbf{F}_p)$ contains an element of order $p^n - 1$.

9. Let K be a field containing a cube root of unity ω and let L/K be a Galois extension with Galois group cyclic of order 3.

(a) Prove that there is $\beta \in L$ such that $\sigma(\beta) = \omega\beta$, where σ is a generator of $\mathrm{Gal}(L/K)$.

(b) Prove that there is $\alpha \in K$ such that $L = K(\sqrt[3]{\alpha})$.