Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

September 4, 2009

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

In the following, $C(X, Y)$ denotes the set of continuous functions from topological spaces X to Y, and $\mathbb{R}$ denotes the real line with the standard topology.

(1) Let $\mathcal{T}$ be the family of subsets of $\mathbb{R}^2$ consisting of the empty set and the complements of finite unions of points and lines. Show that $\mathcal{T}$ is a topology. Is $(X, \mathcal{T})$ Hausdorff?

(2) Let X and Y be two locally compact Hausdorff topological spaces.
  (a) Define the one-point compactification of X.
  (b) Recall that a function is *proper* if the inverse image of any compact set is compact. Prove that a function $f\colon X \to Y$ is proper if and only if it extends to a contiunous map between the one-point compactifications of X and Y.

(3) Let $\{X_i \mid i \in I\}$ be a collection of topological spaces, and for each $i \in I$ let $A_i \subset X_i$. Show that $\prod A_i$ is dense in $\prod X_i$ if and only if each A_i is dense in X_i.

(4) Let X be a set, and let $f_n\colon X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Prove that $\{f_n\}$ converges uniformly to a function $f\colon X \to \mathbb{R}$ if and only if $\{f_n\}$ converges in $(\mathbb{R}^X, \bar{\rho})$.

(5) Let (X, d) be a metric space and $f\colon X \to X$ a function such that $d(x, y) > d(f(x), f(y))$ for all $x \neq y$ in X.
  (a) Give an example to show that if X is not compact, then f need not have any fixed points.
  (b) If X is compact, show that f has a fixed point x_0.
  (c) Show that if f has a fixed point, it is unique.

(6) Let $f\colon X \to Y$ be a continuous bijection.
  (a) Show that if X is compact and Y is Hausdorff, then f is a homeomorphism.
  (b) Give an example where Y is Hausdorff and f is not a homeomorphism.
  (c) Give an example where X is compact and f is not a homeomorphism.

(7) If $A \subseteq X$, a retraction of X onto A is a continuous map $r\colon X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.