DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
SEPTEMBER 2009

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct
PhD: 75% with at least three substantially correct.

1. Use Newton's method to find one root of the function

$$f(x) = x^3 - (2a+2)x^2 + (a^2+4a)x - 2a^2 = (x-2)(x-a)^2.$$

Suppose the initial guess is sufficiently close to $x = 2$.

(a) For which values of a, Newton's method has only the first order convergence? Compute the convergence rate.

(b) For what values of a, Newton's method has the second order convergence?

2. Let A be a real symmetric, positive definite $n \times n$ matrix. After k steps of Gaussian elimination without pivoting, A will be reduced to the form

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{pmatrix},$$

where $A_{22}^{(k)}$ is an $(n-k) \times (n-k)$ matrix. Show by induction that

(a) $A_{22}^{(k)}$ is symmetric and positive definite.

(b) $a_{ii}^{(k)} \le a_{ii}^{(k-1)}$ for $k \le i \le n$, $k = 1, 2, \dots, n-1$.

3. Find the values of a and b which solve the following optimization problem:

$$\min_{a,b} \int_0^\infty (x^2 - ax - b)^2 \, e^{-x} \, dx.$$

Note that the function $f(x) = (ax + b)$ is the weighted L^2 projection of x^2 onto the space spanned by $\{1, x\}$.

4. State the Gauss quadrature for the following integral

$$\int_0^\infty f(x) \, e^{-x} \, dx,$$

and derive the one with exactly two nodes.

5. Let $x_0 < x_1 < \dots < x_n$ be $n + 1$ distinct points.

(a) Prove that there is a unique polynomial of degree at most n that interpolates the function $f(x)$ at these nodes.

(b) Derive the Lagrange form of the interpolation polynomial.

(c) Find the lowest order polynomial $p(x)$ that satisfies the conditions:

$$p(0) = 3, \qquad p(1) = 4, \qquad p'(0) = -1, \qquad p'(1) = 3.$$

6. Consider the PDE $u_t + au_x = 0$ for $x \in [-\pi, \pi]$ with the initial conditions $u(x, 0) = f(x)$ and periodic boundary conditions. The coefficient a is constant. The forward-time central-space scheme is given by

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = 0$$

where Δt discretizes time and $\Delta x = \frac{2\pi}{N}$ discretizes space. Here v_j^n is the approximation to $u(x_j, n\Delta t)$ with $x_j = -\pi + j\Delta x$ for $j = 0, 1, \ldots, N$.

(a) Discuss how the periodic boundary condition can be implemented.

(b) What is the accuracy of the scheme?

(c) For what values of Δt is this scheme stable?

7. Consider the one-step method to approximate the solution of $y' = f(y)$, $y(t_0) = y_0$:

$$\begin{cases} k_1 & = & f(t_n, y_n) \\ k_2 & = & f(t_n + h, y_n + hk_1) \\ y_{n+1} & = & y_n + \frac{1}{2}h(k_1 + k_2) \end{cases}$$

where $h = t_{n+1} - t_n$.

(a) Find a simplified expression for the truncation error of this scheme.

(b) Is the scheme consistent? Explain.