BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. It is a fact that

$$\langle f, g\rangle := \int_{-1}^{1} f(t)g(t)dt$$

defines an *inner product* in $P_3 :=$ the $\mathbf{R}$-vector space of polynomials of degree ≤ 3.

(a) Find an orthonormal basis for the subspace of P_3 spanned by x and x^2.

(b) Complete the basis in Part (a) to an orthonormal basis for P_3.

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2. Let A be an $n \times n$ real matrix. Show that there exists an integer $k > 0$ such that $\big(\text{Null}(A^k)\big) \cap \big(\text{Image}(A^k)\big) = 0$.

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3. Let A be an $n \times n$ real matrix. Show that

$$A^2 = BA \text{ for some invertible matrix } B \Longleftrightarrow \text{rank}(A^2) = \text{rank}(A).$$

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4. Let $A_i x + B_i y + C_i = 0$ $(i = 1, 2, 3)$ be the equations of three lines on the plane $\mathbf{R}^2$. Suppose the lines are pairwise non-parallel. Show that these three lines intersect at one common point if and only if

$$\det\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} = 0.$$

Part II. Advanced Calculus

1. Determine every integer $k > 0$ for which the following function $f : \mathbf{R} \to \mathbf{R}$ is k-th differentiable. Justify your reasoning.

$$f(x) := \begin{cases} x^3 \log |x| & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

2(a). Show that

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \pm \cdots \quad \text{for } |x| < 1.$$

Justify your reasoning.

(b) Show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \cdots .$$

Justify your reasoning.

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3. Find the positively oriented (i.e. counter-clockwise) simple closed curve C on the plane that maximizes the value of the line integral

$$\int_C \Big((y^3 - y)dx - 2x^3 dy \Big).$$

Justify your reasoning.

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4. Among all planes that are tangent to the surface $xy^2z^2 = 1$, find the ones that are farthest from the origin.

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5. Let $g(x), f_n(x)$ $(n = 1, 2, \ldots)$ be continuous functions on $[0, \infty)$. Suppose that

- $|f_n(x)| \leq g(x)$ for all $x \geq 0$ and for all n;
- there exists a function $f(x)$ such that for any fixed, finite $A > 0$, the sequence of functions $f_n(x)$ converge uniformly to $f(x)$ on $[0, A]$; and
- $\displaystyle\int_0^\infty g(x)dx$ exists and is finite.

Show that

$$\lim_{n \to \infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx.$$
