University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Prove the following version of the Frobenius Theorem: Let M be a 2-manifold and X, Y be linearly independent, non-vanishing smooth vector fields defined in a neighborhood of a point $p \in M$ such that $[X, Y] \equiv 0$. Show that there is a smooth chart (x_1, x_2) centered at p such that $X = \frac{\partial}{\partial x_1}$ and $Y = \frac{\partial}{\partial x_2}$.

Problem 2. Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of CP^n. In this problem you may assume the following knowledge about the de Rham cohomology groups of S^n:

$$H^k_{dR}(S^n) = \mathbb{R} \text{ if } k = 0, n, \;\; H^k_{dR}(S^n) = 0 \text{ otherwise,}$$

and the fact that $CP^1 = S^2$.

Problem 3. Let M be a simply-connected, closed smooth 4-manifold, and let β be a closed 3-form on M. Show that there exists a 2-form α on M such that $\beta = d\alpha$.

Problem 4. Let M be a closed, oriented, smooth n-manifold, and $f : M \to M$ be a smooth self-map. A fixed point $x \in M$ of f (i.e., $f(x) = x$) is called *simple* if the map

$$f_{*,x} - Id : T_xM \to T_xM$$

is an isomorphism.

(1) Let $\Delta := \{(x, x)|x \in M\} \subset M \times M$ be the diagonal and $\Gamma_f := \{(x, f(x))|x \in M\} \subset M \times M$ be the graph of f. Show that the submanifolds Δ and Γ_f intersect transversely in $M \times M$ if and only if f has only simple fixed points.

(2) Suppose f has only simple fixed points. Explain how to orient $M \times M$, Δ and Γ_f canonically so that the intersection number

$$\#(\Gamma_f, \Delta) = \sum_{x \text{ is a fixed point of } f} \text{sign}(\det(f_{*,x} - Id)).$$

Problem 5. Let Q be the diagonal matrix with entries 1 and -1, and define $G \subset SL(2, \mathbb{C})$ to be the set of all 2-by-2 complex matrices A with determinant 1 such that $A^*QA = Q$, where A^* is the conjugate transpose of A. .

a) Prove that G is a Lie group, calculate its Lie algebra and compute its dimension.
b) Determine a maximal abelian subalgebra of its Lie alegbra and a corresponding maximal torus in G.

Problem 6. Let M be a smooth n-manifold and let $V(M)$ be the associated bundle of n-frames.

a) Show that $V(M)$ is a smooth principal $GL_n(\mathbb{R})$-bundle over M.

b) Show that $V(M)$ is a trivial bundle if and only if TM is a trivial bundle.

Problem 7. Let $(M, \langle\cdot,\cdot\rangle)$ be a 2-dimensional Riemannian manifold, and let ∇ be the Levi-Civita connection. For any point $x \in M$, define

$$K(x) \equiv \frac{\langle R(X,Y)Y, X\rangle}{\sqrt{|X|^2|Y|^2 - \langle X,Y\rangle^2}},$$

where $X, Y \in T_xM$ is a pair of linearly independent vectors. (Here $R(X,Y)Z \equiv \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature endomorphism.) Show that

(1) $K(x)$ depends only on x (i.e., independent of the choice of X, Y).

(2) If $K \equiv 0$ on M, then $\langle\cdot,\cdot\rangle$ is locally isometric to the Euclidean metric.