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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \mathcal{N})$ a measurable space. Let $T\colon X \to Y$ be a measurable function.

   (a) Define a set function ν on $\mathcal{N}$ by $\nu(A) = \mu(T^{-1}(A))$ for every $A \in \mathcal{N}$. Prove that ν is a measure on $\mathcal{N}$.

   (b) Prove that if $f \in L^1(\nu)$, then $f \circ T \in L^1(\mu)$ and that

   $$\int_Y f \, d\nu = \int_X (f \circ T) \, d\mu.$$

   (c) Consider $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N})$ as in line 1 of this problem and consider the measure ν defined in part (a). Assume that $\mu(X) < \infty$ and that γ is a finite measure on $(Y, \mathcal{N})$ satisfying $\nu \ll \gamma$. By using part (b) and quoting a well known theorem in measure theory, prove that there exists $g \in L^1(\gamma)$ such that

   $$\int_X (f \circ T) \, d\mu = \int_Y f \, g \, d\gamma \quad \text{for each } f \in L^1(\nu).$$

2. Let $\mu = \nu$ be counting measure on the set $\mathbb{N}$ of positive integers; i.e., for $A \subset \mathbb{N}$, $\mu(A) = \nu(A)$ equals the cardinality of A. Define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y \\ -2 + 2^{-x} & \text{if } x = y + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the two iterated integrals

$$\int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(x,y) d\mu(x) \right) d\nu(y) \text{ and } \int_{\mathbb{N}} \left(\int_{\mathbb{N}} f(x,y) d\nu(y) \right) d\mu(x).$$

(a) Prove that both of these iterated integrals exist.

(b) Compute the values of these two iterated integrals.

(c) Expain why your answers to parts (a) and (b) do not contradict the Fubini-Tonelli Theorem.

3. Let X be a Banach space with norm $\|\cdot\|$ and let $L(X,X)$ be the space of all bounded, linear operators mapping X into X.

   (a) For $U \in L(X,X)$ give the definition of $\|U\|$.

   (b) Assume that $U \in L(X,X)$ satisfies $\|I-U\| < 1$, where I is the identity operator. Prove that U is invertible and that $\sum_{n=0}^{\infty}(I-U)^n$ converges in $L(X,X)$ to U^{-1}.

   (c) Assume that $U \in L(X,X)$ is invertible and that $W \in L(X,X)$ satisfies $\|W-U\| < \|U^{-1}\|^{-1}$. Prove that W is invertible.

4. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let u and v be linearly independent, unit vectors in H. Define M to be the linear span of u and v.

   (a) Determine a unit vector w such that $\langle u, w \rangle = 0$ and the linear span of u and w equals M. Be sure that you verify the latter statement about the linear span of u and w.

   (b) Let x be an element in $H \setminus M$. Determine explicitly, in terms of u and w, a $y_0 \in M$ such that

   $$\|x - y_0\| = \inf\{\|x - z\| : z \in M\}.$$

   (c) Prove that the y_0 found in part (b) is unique and re-express it in terms of u and v.

5. Let m be Lebesgue measure on $\mathbb{R}$. Let f be a bounded, measurable function mapping $\mathbb{R}$ into $[0, \infty)$ and satisfying

$$\int_{\mathbb{R}} f \, dm = K \ \text{ for some } 0 < K < \infty.$$

Compute

$$\lim_{n \to \infty} \int_{\mathbb{R}} n \, \log \left[1 + \left(\frac{f(x)}{n} \right)^{\alpha} \right] dm$$

for constant $\alpha > 0$ in the following three cases:

(a) $0 < \alpha < 1$,

(b) $\alpha = 1$,

(c) $1 < \alpha < \infty$.

(**Hint.** Use Fatou's Lemma in part (a).)

6. Let H be a Hilbert space. Recall that if M is a closed subspace of H, then one can define a linear operator $P_M : H \to H$ by defining $P_M x$ to be the element of M such that $x - P_M x \in M^\perp$. P_M is called the orthogonal projection of H onto M.

   Assume that there exists a sequence $\{M_n, n \in \mathbb{N}\}$ of closed subspaces of H such that for all $n \in \mathbb{N}$ we have $M_n \subset M_{n+1}$. In this case we define

$$Y = \overline{\cup_{n=1}^{\infty} M_n}\,;$$

   i.e., Y equals the closure of the union of all the subspaces M_n.

   (a) Prove that Y is a closed subspace of H.

   (b) Prove that for all $x \in H$, $\|x - P_{M_n} x\| \to \|x - P_Y x\|$ as $n \to \infty$. The linear operator P_Y is well defined because of part (a).

   (c) Prove that for all $x \in H$, $P_{M_n} x$ converges to $P_Y x$ as $n \to \infty$. (**Hint.** Use part (a) and the Pythagorean Theorem.)

7. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Let $K(x, y)$ be a measurable function mapping $X \times Y$ into $\mathbb{R}$ with the following property. There exists a finite constant $M > 0$ such that for μ-almost every x

$$\int_Y |K(x,y)|\, d\nu(y) \leq M$$

and for ν-almost every y

$$\int_X |K(x,y)|\, d\mu(x) \leq M.$$

Prove that the operator

$$T : f \mapsto \int_{X \times Y} K(x,y)\, f(y)\, d\nu(y)$$

is a bounded operator from $L^p(Y)$ into $L^p(X)$ for all $1 \leq p \leq \infty$. Also prove that the operator norm of T does not exceed M. (**Hint.** For $1 < p < \infty$ first compute a suitable bound on $|Tf(x)|$ by applying Hölder's inequality to an appropriate factorization of the integrand.)

8. Let m be Lebesgue measure on $\mathbb{R}$. Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of measurable functions on $\mathbb{R}$ and let f be a measurable function on $\mathbb{R}$.

   (a) Define the concept that the sequence $\{f_n, n \in \mathbb{N}\}$ converges to f in measure.

   (b) Assume that the sequence $\{f_n, n \in \mathbb{N}\}$ converges to f in measure. Prove that there exists a subsequence $\{f_{n_k}, k \in \mathbb{N}\}$ that converges to f almost everywhere. (**Hint.** Prove that there exist positive integers $1 \leq n_1 < n_2 \ldots < n_k < \ldots$ such that for all $k \in \mathbb{N}$

$$m\{x \in \mathbb{R} : |f_{n_k}(x) - f(x)| \geq 2^{-k}\} \leq 2^{-k}.)$$