DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

WEDNESDAY, SEPTEMBER 2, 2009

Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the FOUR parts.

Part I.

1. For any finite group G and any prime p, denote by $n_p(G)$ the number of Sylow p-subgroup of G. If N is a normal subgroup of G, show that $n_p(G/N) \leq n_p(G)$.

2. Let p be a prime, and let G be the group $\mathbf{Z}/p \times \mathbf{Z}/p^2 \times \mathbf{Z}/p^3$.
(a) Determine the number of *cyclic* subgroup of G of order p^2. Justify your reasoning.
(b) Determine the number of subgroup (not necessarily cyclic) of G of order p^3. Justify your reasoning.

Part II.

1. Let R be a commutative ring with $1 \neq 0$. Denote by $R[x]$ the one-variable polynomial ring over R. Fix an element $f = a_n x^n + \cdots + a_1 x + a_0 \in R[x]$.
(a) Show that f is a unit in $R[x]$ if and only if a_0 is a unit in R and that the remaining a_i are nilpotent.
(b) Show that f is nilpotent if and only if all a_i are nilpotent.

2(a) Prove or give a counterexample: the quotient ring of a PID by a prime ideal is a PID.
(b) Prove or give a counterexample: the quotient ring of a UFD by a prime ideal is a UFD.

Part III.

1. Let $f_1, f_2 \in K[x]$ be non-constant polynomials over a field K.
(a) Determine $\dfrac{K[x]}{(f_1)} \otimes_{K[x]} \dfrac{K[x]}{(f_2)}$ as a $K[x]$-module. Show your work.
(b) Determine $\dfrac{K[x]}{(f_1)} \otimes_{K} \dfrac{K[x]}{(f_2)}$ as a K-module. Show your work.

2. Let p be a prime, and let $\mathbf{F}_q$ be a finite field with $q = p^n$ elements. Denote by $\pi_q : \mathbf{F}_q \to \mathbf{F}_q$ the Frobenius automorphism given by $\alpha \mapsto \alpha^p$.
(a) Determine the dimension of $\mathbf{F}_q$ as a $\mathbf{F}_p$-vector space.
(b) Show that as a $\mathbf{F}_p$-linear map, π_q is diagonalizable if and only if n divides $p - 1$.

Part IV.

1(a) Let K/k be a finite field extension. Let R be a ring such that $k \subset R \subset K$. Show that R is a field.
(b) Give a counterexample to show that Part (a) is false in general if K/k is not a finite extension. Justify your reasoning.

2. Show that if the Galois group of a cubic polynomial over $\mathbf{Q}$ is cyclic of order 3, then all three roots of this polynomials are real.