DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
Wednesday, August 27, 2008

Work all five problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (bf 20 pts) Give a precise **definition** for the following:
   (a) A complete family of density functions.
   (b) A Uniformly Most Powerful (UMP) test.
   (c) A regular exponential class of density functions.
   (d) State the Lehmann-Scheffe Theorem.

2. (**15** pts) The p.d.f. of an exponential distribution with mean θ is

$$f(x) = \theta^{-1} \exp(-x/\theta) \quad \text{for } x > 0, \text{ and } 0 \text{ elsewhere.}$$

   Let $X_1, \ldots X_n$ be a random sample from this p.d.f.
   (a) Derive the MLE of θ. It is required to justify that your answer is indeed an MLE.
   (b) Give the MLE of θ^2, with justification (note that $\theta^2 = Var(X_i)$).
   (c) For a large n, find an approximate 95% confidence interval for θ and for θ^2, respectively.

3. (**20** pts) Let $X_1, \ldots, X_n$ be a random sample form an exponential distribution with density $f(x; \lambda) = \lambda e^{-\lambda x}$, $x > 0$ (having mean $1/\lambda$). Assume a prior density for λ which is also exponential with mean $1/\beta$, where β is known.
   (a) Prove that the posterior distribution of β is a Gamma distribution. *If you can't do part (a), assume the posterior distribution is Gamma with parameters a and b and do the remaining parts.*
   (b) Using squared error loss find the Bayes estimator of λ.
   (c) Derive a 95% Bayesian confidence interval for λ.

4. (**25** pts) Suppose that $X_1, \ldots, X_n$ is a random sample from a $N(\mu, \sigma^2)$ distribution, with μ and σ^2 unknown.

   (a) Write down, without proof, the MLEs of μ and σ^2, respectively.

   (b) Write down, without proof, the MLE of σ^2 given $\mu = \mu_0$.

   (c) Using the given sample, derive an α-level likelihood ratio test for $H_0 : \ \mu = \mu_0$ against the alternative $H_1 : \ \mu \neq \mu_0$, where μ_0 is a given number.

   (d) For a large n and $\alpha = 0.05$, find the asymptotic power of the test if $\mu = \mu_0 + 1$. You may use $\Phi(\cdot)$ to denote the c.d.f. of the $N(0, 1)$ distribution.

5. (**20** pts)Let $X_1, \cdots, X_{25}$ be a random sample from a normal distribution with an unknown mean μ and variance 1. Consider testing the hypotheses

$$H_0 : \ \mu \leq 0 \ \text{ against } \ H_1 : \ \mu > 0.$$

   It is known that the UMP size 0.05 test rejects H_0 iff $5\bar{X} > 1.645$.

   (a) Explain what it means for the test to have size 0.05, and what it means to be UMP.

   (b) Construct the power function of the test, and calculate the power of the test at $\mu = 0.5$. Is the power function an increasing function of μ? (Explain.)

   (c) What is the Type I error probability of the test at $\mu = 0$? Is this probability larger or smaller than the Type I error probability at $\mu = -1$? (Explain briefly.)