DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - NUMERICS
August, 2008

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. Find the Padé approximation

$$R_{1,1}(x) \equiv \frac{a+bx}{1+cx} = p_2(x) + O(x^3)$$

of the polynomial $p_2(x) = 1 - \frac{1}{2}\,x + \frac{1}{24}\,x^2$, and use it to deduce the approximation

$$\cos(x) = \frac{12-5\,x^2}{12+x^2} + O(x^6).$$

2. Consider the fixed point iteration defined by the formula $x_{n+1} = F(x_n)$, where

$$F(x) = x - a + 2ae^{-x}.$$

Here $a \neq 0$ is a parameter.

(a) Find the fixed point, p.

(b) Determine for which values of a the iteration converges to p and those for which they diverge away from p. Your answer must be in terms of intervals or unions of intervals.

(c) Does there exist a value of a for which the iteration converges quadratically? If so, find it.

3. Consider the numerical integration rule

$$I(f) = \int_{-h}^{h} f(x)\,dx \approx A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2).$$

(a) Find A_0, A_1, and A_2 such that the integration rule is exact for polynomials of degree ≤ 2.

(b) Show that the rule constructed in (a) is in fact exact for polynomials of degree ≤ 3.

(c) For the constructed rule, it can be proved that

$$I(f) - [A_0 f(-h/2) + A_1 f(0 + A_2 f(h/2)] = c_0 f^{(4)}(\eta) h^5, \;\; \eta \in (-h, h)$$

where c_0 is a constant independent of f. Find the constant c_0.

4. Consider the ODE system

$$u_t = -Au,$$

where A is a constant symmetric positive definite matrix.

(a) Construct a fourth-order numerical scheme for the above system.

(b) Give the stability condition for this scheme.

(Hint: First consider a similarity transformation of A.)

5. An $n \times n$ matrix of the form $N(\mathbf{y}, k) = I - \mathbf{y}\mathbf{e_k^T}$ is called a *Gauss-Jordan matrix.* Here $\mathbf{e_k}$ is the kth unit coordinate vector, and $\mathbf{y}$ is an arbitrary vector.

(a) Find a formula for $N(\mathbf{y}, k)^{-1}$. Under what conditions does this inverse exist?

(b) Let $\mathbf{x}$ be an arbitrary vector. For a given k, find a vector $\mathbf{y}$ so that $N(\mathbf{y}, k)\mathbf{x} = \mathbf{e_k}$. Under what conditions does such a $\mathbf{y}$ exist?

(c) For a given square matrix A, give an algorithm based on Gauss-Jordan matrices that computes A^{-1}. Under what conditions will this algorithm work?

6. Consider the difference scheme

$$y_{n+1} = \alpha\, y_{n-1} + \beta\, y_n + \gamma\, h\, f(x_n, y_n)$$

for approximating the solution to the equation

$$\frac{dy}{dx} = f\left(x, y(x)\right).$$

Find constants α, β and γ for which this has highest order, and the corresponding local truncation error.

7. Given a function $f(x)$, and a set of Gauss quadrature points and weights: $\{x_i\}_{i=1}^n$, $\{\omega_i\}_{i=1}^n$. Let $P_k(x)$ be the Legendre polynomial of degree k.

   (a) Define the interpolation of $f(x)$ on the space $span\{P_0(x), P_1(x), \ldots, P_{n-1}(x)\}$, with interpolation points $\{x_i\}_{i=1}^n$

   (b) Find the L^2 projection of $f(x)$ onto the above space while using the given Gauss quadrature to evaluate the integrals.

   (c) Are the two approximate functions obtained in (a) and (b) the same? Give a detailed explanation.