DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
August 2008

Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. A model for competition between 2 species is given by

$$\begin{aligned}\frac{dh}{dt} &= h(1-h-p)\\ \frac{dp}{dt} &= p(\frac{1}{2}-\frac{1}{4}p-\frac{3}{4}h)\end{aligned}$$

   where $h(t)$ and $p(t)$ represent the densities at time t.

   (a) Identify all of the critical points.
   (b) Find the corresponding linear system for each critical point.
   (c) Determine the stability of each critical point.
   (d) Sketch the phase portrait.

2. Consider the partial differential equation

$$u_x + yu_y = y$$

   (a) Sketch the characteristics.
   (b) Given that $u(1,1) = 1$, what value of $u(3,2)$ is obtained by following the characteristic?

3. Consider an infinite string whose position is governed by the 1-D wave equation

$$u_{tt} = u_{xx}$$

   The string's initial position $u(x,0) = 0$ and initial velocity $u_t(x,0) = 1$ for $|x| < a$ and $u_t(x,0) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2, 3a/2$ and $5a$.

4. Consider the partial differential equation

$$u_{xx} + 4u_x - 2u_t + 8u = 0$$

with initial condition $u(x,0) = e^{-2t+2x}\sin(4\pi x)$ and boundary conditions $u(0,t) = u(1,t) = 0$.

(a) Use the change of variables $u(x,t) = e^{ax+bt}w(x,t)$ to convert the above equation into a diffusion equation for w. a and b must be chosen appropriately.

(b) Give the boundary and initial conditions of w.

(c) Solve the diffusion equation for w then give the solution u.

5. The following model was proposed for the evolution of an epidemic:

$$\begin{aligned} \frac{dx}{dt} &= -kxy \\ \frac{dy}{dt} &= kxy - Ly \\ \frac{dz}{dt} &= Ly \end{aligned}$$

The population is divided into 3 classes: $x(t)$ = number of health people; $y(t)$ = number of sick people; $z(t)$ = number of dead people. k and L are postive constants.

(a) Show that $x(t)+y(t)+z(t) = N$, where N is a constant. Interpret this result.

(b) Use the $\frac{dx}{dt}$ and $\frac{dz}{dt}$ equations to show that

$$x(t) = x_0 e^{(-kz(t)/L)}$$

where $x_0 = x(0)$.

(c) Verify that

$$\frac{dz}{dt} = L[N - z - x_0 e^{(-kz/L)}].$$

(d) Using a geometrical argument, how many fixed points does the above equation have and determine the stability of the fixed points(Hint: look at the graphs of the functions $N - z$ and $x_0 e^{(-kz/L)}$). What does this mean physically as far as the evolution of the number of dead people?

6. The Maxwell-Bloch equations provide a model for a laser. These equations describe the dynamics of the electric field E, the mean polarization P of the atoms, and the population inversion D:

$$\begin{aligned} \frac{dE}{dt} &= \kappa(P - E) \\ \frac{dP}{dt} &= \gamma_1(ED - P) \\ \frac{dD}{dt} &= \gamma_2(\lambda + 1 - D - \lambda EP) \end{aligned}$$

where κ is the decay rate in the laser cavity due to beam transmission, γ_1 and γ_2 are the decay rates of the atomic polarization and population inversion, respectively, and λ is the pumping energy parameter. The parameter λ may be any real number. All of the other parameters are positive.

(a) Assuming $\frac{dP}{dt} = 0$ and $\frac{dD}{dt} = 0$, express P and D in terms of E, and thereby derive a first-order equation for the evolution of E.

(b) Find all the fixed points of the equation for E.

(c) Draw the bifurcation diagram of E^* versus λ(Be sure to distinguish between stable and unstable branches. E^* is the equilibrium solutions.