Your Name :

Department of Mathematics and Statistics
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Advanced Qualifying Exam– Differential Equations.

Monday August 25th, 2008

10am to 1pm – LGRT 1234

This exam consists of seven (7) problems all carrying equal weight. You must do five (5) of them. Passing level: 75% with at least three (3) substantially complete solutions. Please **justify** all your steps properly by indicating (or stating) the result you are using. Please write each problem clearly and neatly in a separate page.

(**1**) Consider the initial value problem

$$\begin{cases} u_{tt} - c^2\, u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = p(x) & \\ u_t(x,0) = q(x) & \end{cases}$$

where $p(x), q(x)$ are known given smooth functions and $c > 0$.

(a) By direct calculation show that the D'Alembert formula gives the solution to the problem above; *i.e* verify all identities above are satisfied.

(b) Denote by χ the function of one variable

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $p(x) = \chi(x)$ and $q(x) = 0$. Show then that for each fixed $t > 0$, the solution $u(x,t)$ obtained explicitly from the D'Alembert formula vanishes on the two half intervals (for t fixed):

$$x \in (-\infty, -(1+ct)) \qquad x \in (1+ct, \infty)$$

(**2**) Let $f(x)$ be the vector field on $x \in \mathbb{R}^2$

$$f(x) = \binom{H_{x_2}(x)}{-H_{x_1}(x)} \qquad\qquad x = (x_1, x_2)$$

where $H(x)$ is a given smooth function of x; and let $x(t,\xi)$ be the solution of

$$\frac{dx}{dt} = f(x), \qquad\qquad x(0,\xi) = \xi$$

for each initial condition in $\xi \in \mathbb{R}^2$.

Let $B_0 = \{\xi \,:\, \xi_1^2 + \xi_2^2 \leq 1\}$ and define $B(t) = \{x(t,\xi) : \xi \in B_0\}$. Show that

$$\int\int_{B(t)} dx = \pi \qquad\qquad \text{for all } t,$$

i.e., the area of $B(t)$ is constant in time. (*Hint: Consider the (linearized) system of ODE's satisfied by the Jacobian $x_\xi(t,\xi)$ of the solution of the original initial value problem with respect to the initial conditions.*)

(**3**) (a) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. (Recall a *domain* is a connected open subset of $\mathbb{R}^n$)

Assume that

$$u \in C^2(\Omega \times (0,\infty)) \cap C^1(\overline{\Omega} \times [0,\infty))$$

is a solution of

$$\begin{cases} u_{tt} = c^2 \Delta u \\ u(x,t) = 0 \quad \text{for } x \in \partial\Omega,\ t \geq 0, \end{cases}$$

where c is a constant.

Show that the energy

$$E(t) := \frac{1}{2} \int_\Omega \left(u_t^2 + c^2 \, |\nabla u|^2 \right) dx$$

is conserved; *i.e.* $E(t) = E(0)$ for all $t \geq 0$.

(b) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Use part (a) to show uniqueness of the solution for the non-homogeneous boundary/initial value problem wave

$$\begin{cases} u_{tt} - \Delta u = f(x,t) \qquad x \in \Omega, t > 0 \\ u(x,t) = g(x,t) \quad \text{on } \partial\Omega, t \geq 0 \\ u(x,0) = h(x), \quad u_t(x,0) = k(x) \quad x \in \Omega \end{cases}$$

where f, g, h, k are given smooth functions and any solution u is assumed to belong to the space $C^2(\Omega \times (0,\infty)) \cap C^1(\overline{\Omega} \times [0,\infty))$

(**4**) Consider the 1-parameter family of equations

$$(*)\begin{cases} x' &= y^3 - y - x \\ y' &= x - A \end{cases}$$

(a) When $A = 0$ give a complete and rigorous analysis of the global behavior of all solutions to $(*)$, paying particular attention the behaviors of solutions near rest points, and the presence of any periodic, homoclinic, or heteroclinic orbits, if any such orbits exist. A phase plane sketch of the various behaviors that occur must be justified appropriate analytical calculations and arguments. Useful techniques include linearization, invariant regions, Liapunov functions, and the stable/unstable manifolds.

(b) How would you expect the phase planes of $(*)$ to change as A is increased from $A = 0$ to $A = 1$? Your answer may be left in the form of a conjecture obtained from information you are able to calculate for particular values of A; a complete and rigorous justification is not required.

(**5**) Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain with smooth boundary. Let $G(x, y)$ denote the Green's function for this domain. We know that for fixed $x \in \Omega$

$$G(x, y) := K(y - x) - \omega_x(y)$$

where $\omega_x(y)$ satisfies

$$\begin{cases} \Delta_y \, \omega_x \; = \; 0 \qquad \text{in } \; \Omega, \\ \omega_x(y) \; = \; K(y - x) \qquad \text{for } \; y \in \partial\Omega \end{cases}$$

and

$$K(x) := \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}, \qquad \alpha(n) = \text{ volume of unit ball in } \mathbb{R}^n$$

for $x \in \mathbb{R}^n$, $x \neq 0$ is the fundamental solution of the Laplace's equation.

Thus $y \to G(x, y)$ is harmonic for $y \in \Omega, y \neq x$ and $G(x, y) = 0$ for $y \in \partial\Omega$. Moreover, $G(x, y) = G(y, x)$ (you do not need to prove this).

Use the maximum/minimum principle to show that $G(x, y) > 0$ for all $x, y \in \Omega$ with $x \neq y$.

(**6**) (a) Determine the variation of parameters solution to

$$x'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \qquad x(0) = \begin{pmatrix} a \\ b \end{pmatrix}$$

where $f_1(t)$ and $f_2(t)$ are continuous and a, b are constants.

(b) Determine the first two coefficients $X_0(t)$ and $X_1(t)$ in the Taylor expansion

$$\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} X_n(t)$$

of the exact solution $x(t,\varepsilon)$ of

$$x' = \begin{pmatrix} 0 & -1 \\ 1 & \varepsilon \sin t \cos t \end{pmatrix} x \qquad x(0) = \begin{pmatrix} a \\ b \end{pmatrix} \qquad (\dagger)$$

(c) Viewing (†) as a linear periodic system with a 2π-periodic coefficient matrix, use the first order approximation $x_1(t,\varepsilon) = X_0(t) + \varepsilon X_1(t)$ to the exact solution $x(t,\varepsilon)$ of (†) to obtain an approximation to the system's Floquet multipliers, and use this calculation to formulate a conjecture about the asymptotic stability of the exact solution $x(t,\varepsilon)$ for sufficiently small $\varepsilon \neq 0$. (*Hints: Use $x_1(t,\varepsilon)$ to approximate each of the columns of the fundamental matrix solution $\Phi(t,\varepsilon)$ with $\Phi(0,\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for small ε; also, $\int_0^{2\pi} \sin^2 t \cos^2 t \, dt = \pi/4$.*)

(**7**) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Define

$$\lambda_1 = \lambda_1(\Omega) := \inf_{u \in C_c^\infty(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx}$$

where $C_c^\infty(\Omega)$ is the set of C^∞ functions whose support is a compact subset in Ω.

(a) Prove the Poincaré inequality: *i.e* for $u \in C_c^\infty(\Omega)$ and $1 \leq p < \infty$

$$\|u\|_{L^p} \leq C \, \|\nabla u\|_{L^p}$$

where $C > 0$ is a constant depending possibly on Ω and p but independent of u. Deduce from it that $\lambda_1 > 0$. (*Hints. Assume $\Omega \subset [-M, M]^n$; write $u(x) = \frac{1}{2}\{\int_{-M}^{x} \partial_1 u(y, x_2, \dots x_n) \, dy - \int_x^M \partial_1 u(y, x_2, \dots x_n) \, dy\}$ (why?)*).

(b) Prove that for all $f \in L^2(\Omega)$ and for all constants $\gamma > -\lambda_1$, there exists a weak solution $u \in H_0^1(\Omega)$ of

$$\begin{cases} -\Delta u + \gamma \, u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$