Department of Mathematics and Statistics, UMass–Amherst
Advanced Exam – Algebra, August 29, 2008

Three hours. In order to pass, you must score 65% and have at least one problem substantially correct from each part. Partial credit will be awarded. Be sure to explain each of your answers. All rings have a multiplicative identity 1.

Part I. Group Theory

1. Let $\mathrm{Aut}(G)$ denote the automorphism group of a group G.
   (a) Show that $\mathrm{Aut}(G)$ is in fact a group.
   (b) Describe $\mathrm{Aut}(G)$ when G is a cyclic group of order p, a prime number.
   (c) Let G be finite. Show that if $\mathrm{Aut}(G)$ is cyclic, then G is abelian. Hint: consider the natural homomorphism $G \to \mathrm{Aut}(G)$.

2. Show that there are at most 5 isomorphism classes of groups of order 20.

Part II. Rings and Modules

3. A chain of prime ideals of length n in a commutative ring R is an increasing sequence
$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n \subsetneq R,$$
where P_i is a prime ideal in R.
   (a) Show that if R is a PID, every chain of prime ideals has length 0 or 1.
   (b) Exhibit a chain of prime ideals of length 2 in $\mathbb{Z}[x]$.
   (c) Find a ring R with a chain of prime ideals of length 2008.

4. Let $I = (2, x)$ be the ideal generated by 2 and x in $R = \mathbb{Z}[x]$. Note that $R/I \cong \mathbb{Z}/2\mathbb{Z}$, so the latter is naturally an R-module.
   (a) Show that the map $\phi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by
$$\phi(a_0 + a_1x + \cdots + a_nx^n, b_0 + b_1x + \cdots + b_mx^m) = \frac{a_0b_1}{2} \bmod 2$$
is R-bilinear and conclude that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ which sends the pure tensor
$$(a_0 + a_1x + \cdots + a_nx^n) \otimes (b_0 + b_1x + \cdots + b_mx^m)$$
to $\frac{a_0b_1}{2}$ mod 2.
   (b) Show that $2 \otimes x - x \otimes 2$ is nonzero in $I \otimes_R I$.

5. Suppose $A, B \in \mathrm{M}_n(\mathbb{R})$ are conjugate by a matrix in $\mathrm{GL}_n(\mathbb{C})$. Show that they are conjugate by a matrix in $\mathrm{GL}_n(\mathbb{R})$.

6. The exponential $\exp(A)$ of a complex matrix A is defined by the power series:

$$\exp(A) = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots + \frac{A^k}{k!} + \ldots.$$

This series always converge, for any complex matrix.

(a) Using the Jordan canonical form, or otherwise, show that

$$\det(\exp(A)) = e^{\operatorname{tr}(A)},$$

where $\operatorname{tr}(A)$ is the trace of A (the sum of the diagonal entries of A).

(b) Compute $\exp(A)$ for

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}.$$

Part III. Galois Theory

7. Let $F \subset E$ be fields with $E = F(\alpha)$. If E/F has odd degree, show that $F(\alpha^2) = F(\alpha)$.

8. Construct the subfields of $F = \mathbb{Q}(\zeta_{19})$, where $\zeta_{19} \neq 1$ satisfies $(\zeta_{19})^{19} = 1$. That is, write each subfield in the form $\mathbb{Q}(\alpha)$ for some $\alpha \in F$.