Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

August 31, 2007

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

Let $C(X, Y)$ denote the set of continuous functions from topological spaces X to Y. Let $\mathbb{R}$ denote the real line with the standard topology.

(1) (a) Give an example of a space which is Hausdorff, locally-connected, and not connected.
(b) Give an example of a space which is Hausdorff, path-connected, and not locally path-connected.

(2) Consider the sequence $f_n(x) = \sin(\frac{x}{n})$. Determine whether or not the sequence converges in each of the following topologies of $C(\mathbb{R}, \mathbb{R})$: the uniform topology, the compact-open topology, and the point-open (pointwise convergence) topology.

(3) Let $X = \mathbb{R}/\mathbb{Z}$ be the quotient space where the integers $\mathbb{Z} \subset \mathbb{R}$ are identified to a single point. Prove that X is connected, Hausdorff, and non-compact.

(4) Prove the Uniform Limit Theorem: "Let X be any topological space and Y a metric space. Let $f_n \in C(X, Y)$ be a sequence which converges uniformly to f. Then $f \in C(X, Y)$."

(5) (a) Let $f : X \to Y$ be a quotient map, where Y is connected. Suppose that for all $y \in Y$, $f^{-1}(y)$ is connected. Show that X is connected.
(b) Let $g : A \to B$ be continuous, where B is path-connected. Suppose that for all $b \in B$, $g^{-1}(b)$ is path-connected. Suppose there exists $h \in C(B, A)$ such that $g \circ h$ is the identity on B. Show that A is path-connected.

(6) Let $\mathbb{R}_d$ and $\mathbb{R}_l$ denote the the real line with the the discrete topology and the lower limit topology, respectively. Recall that a basis for $\mathbb{R}_l$ is the set of intervals $[a, b)$ where $a < b$. List the functions that make up the following sets: $C(\mathbb{R}, \mathbb{R}_d), C(\mathbb{R}_d, \mathbb{R}_l)$, $C(\mathbb{R}, \mathbb{R}_l)$.

(7) Consider $\mathbb{R}^\omega$ with the product topology, the box topology and the uniform topology. Define the subset

$$A = \{(x_1, x_2, \ldots) \in \mathbb{R}^\omega \mid 0 < x_i < 1 \text{ for all } i \in \mathbb{N}\}.$$

In each topology, describe whether or not A is open.