DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERICS
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Do five of the following problems. All problems carry equal weight.
Passing Level:
Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. Which of the following iterations will converge to the indicated fixed point α(provided x_0 is sufficiently close to α)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

   (a) $x_{n+1} = -1 + x_n + \frac{2}{x_n}, \quad \alpha = 2.$
   (b) $x_{n+1} = \frac{1}{2}x_n + \frac{3}{2x_n}, \quad \alpha = 3^{1/3}.$
   (c) $x_{n+1} = \frac{16}{1+x_n} - 1, \quad \alpha = 3.$

2. Determine values of α in the matrix below

$$\begin{pmatrix} 9 & 6 & 9 \\ 6 & 20 & 26 \\ 9 & 26 & \alpha \end{pmatrix}$$

   for which the matrix is positive definite.(Hint: Use Cholesky Decomposition.)

3. Solve the following minimization problem and determine whether there is a unique α that gives the minimum. α is allowed to range over all reals.

$$\min_{\alpha} \int_{-1}^{1} (x - \alpha x^2)^2 dx$$

   We are approximating $f(x) = x$ by polynomials of the form αx^2 in the continuous least squares sense.

4. Consider a well-posed system of ordinary differentiation equations

$$\begin{cases} u_t = Au \\ u(0) = u^0, \end{cases}$$

where A is a constant matrix and u^0 is the initial condition.

   (a) Write down the forward Euler scheme for such a system.

   (b) Give a non-trivial condition for Δt which guarantees stability.

   (c) For what class of matrices A is the condition $\rho(A)\Delta t < 1$, where $\rho(A)$ is the *spectral radius* of A, is also a sufficient condition for the stability?

5. An $n \times n$ matrix A is said to be *strictly diagonally dominant* if

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}| \qquad \text{for } i = 1, \ldots, n\,.$$

Note that the strict inequality implies that each diagonal entry a_{ii} is non-zero. Prove that the Jacobi iteration matrix B_J for A satisfies $||B_J||_\infty < 1$ and therefore the iteration converges in this case for any initial vector $x^{(0)}$.

6. Consider the integral $\int_{-1}^{1} f(x)\, dx$,

   (a) Give the definition of Gauss quadrature

   (b) Find the points and weights for the Gauss quadrature with exactly two points.

7. Consider the ODE initial-value problem

$$\frac{dy}{dx}(x) = f(x, y(x)),$$

with initial data $y(x_0) = y_0$. We would like to solve this initial value problem at points $x_n = nh, n = 0, ..., N$ where $h = x_n - x_{n-1}$ for all n. Find the highest order method in the class

$$y_{n+1} = y_n + h[b_1 f(x_n, y_n) + b_2 f(x_{n-1}, y_{n-1})].$$

i.e., find b_1 and b_2 for the above method which gives the highest order local truncation error. State the order of the method obtained.