Department of Mathematics and Statistics
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Basic Exam: Topology

September 1, 2006

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Recall that X/A denotes the quotient of the space X where the subspace A is identified to a point. Let $Z = X/Y$, where X is the closed interval $[0,1]$, and Y is the open interval $(1/2, 3/4)$. Show that Z is connected, compact, but not Hausdorff.

(2) (a) Let X be a metric space, and let $A \subset X$ be a subspace. If A (taken with the induced metric) is complete, show that A is closed in X.
(b) Show that a compact metric space is complete.

(3) A space is called *totally disconnected* if the only connected subsets are single points.
(a) Suppose $\{X_\alpha\}_{\alpha \in A}$ is a family of totally disconnected spaces. Show that

$$X_A := \prod_{\alpha \in A} X_\alpha,$$

equipped with the product topology, is totally disconnected.
(b) Let A be a countably infinite set, and let $X_\alpha \simeq \{0,1\}$ (a two-point space with the discrete topology) for each $\alpha \in A$. Show that X_A is not a discrete space.

(4) (a) Give the definition of the one-point compactification of a locally compact Hausdorff space X, and show directly from the definition that it is compact and Hausdorff.
(b) Prove that the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1.

(5) Let X be a topological space, and let $F\colon \mathbb{R} \times X \to \mathbb{R}$ be a continuous function. For $t \in \mathbb{R}$, define $f_t\colon X \to \mathbb{R}$ by $f_t(x) = F(x,t)$.
(a) If X is compact, show that $f_{1/n}$ converges uniformly to f_0.
(b) Show by example that this need not be true if X is not compact.

(over)

(6) Call a subset $S \subset \mathbb{R}^2$ "step-connected" if any two points $x, y \in S$ are connected by a "stair-step" path: i.e. there exists a continuous function $f\colon [0,1] \to S$ so that $f(t_0) = x$, $f(t_n) = y$, and there is a subdivision $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ for which f restricted to each interval $[t_i, t_{i+1}]$ is either horizontal or vertical, meaning that either the first coordinate or the second coordinate of f is constant on this interval.

Show that an open connected subset of $\mathbb{R}^2$ is step-connected.

(7) Consider the product, uniform and box topologies in $\mathbb{R}^\omega$. The uniform topology is the topology induced by the metric

$$d(\,(x_1, x_2, \dots), (y_1, y_2, \dots)\,) = \sup_n(\min(1, |x_n - y_n|))\,).$$

The box topology is the topology generated by the basis consisting of all sets

$$U_1 \times U_2 \times U_3 \times \cdots \times U_n \times \cdots$$

where each U_i is an open set in $\mathbb{R}$.

In which of these topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \dots) \\ g(t) &= (t, t, t, \dots) \\ h(t) &= (t, \frac{t}{2}, \frac{t}{3}, \dots) \end{aligned}$$