DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
September 1, 2006 (3 Hours)

Note: *There are five problems with full mark of 100 points. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.*

1. (20 points) Let $X_1, \dots X_n$ be a random sample from a geometric distribution with density $f(x) = P(X_i = x) = \theta(1-\theta)^{x-1}$, for $x = 1, 2, \dots$, where $0 \le \theta \le 1$. The mean of this distribution is $1/\theta$ and the variance is $\sigma^2 = (1-\theta)/\theta^2$.

   (a) Find the maximum likelihood estimator of θ; denote it by $\hat{\theta}$. Be sure to verify that you have maximized the likelihood and discuss whether there is ever an issue with the existence of the MLE.

   (b) Find the Cramer-Rao lower bound for unbiased estimators of θ, and then give the approximate distribution of $\hat{\theta}$ as n gets large.

   (c) Use the result in part (b) to develop an approximate confidence interval for θ.

   (d) The population coefficient of variation is $\gamma = \sigma/\mu = (1-\theta)^{1/2}$. Find the MLE for γ, obtain their approximate large sample distribution of the MLE and develop an approximate confidence interval for γ.

2. (20 points) Suppose that $X_1, \cdots, X_n$ are iid Bernoulli(p) where $0 < p < 1$ is an unknown parameter with $n \ge 2$. Let $T = \sum_{i=1}^n X_i$ and $\tau(p) = p^2$.

   (a) Show that T is a complete and sufficient statistic.

   (b) Show that $X_1 X_2$ is an unbiased estimator for $\tau(p)$.

   (c) Find $E(X_1X_2|T)$ (in an explicit form). *(Hint: $E(X_1X_2|T) = P(X_1X_2 = 1|T)$)*

   (d) Show that $E(X_1X_2|T)$ is the UMVUE for $\tau(p)$. State the name of the theorem used.

3. (10 points) Suppose that $X_1, \cdots, X_n$ are iid Poisson(θ) where $\theta > 0$ is an unknown population mean. Assume the prior density $h(\theta) = e^{-\theta} I(\theta > 0)$. Derive the posterior pdf of θ given that $T = t$ where $T = \sum_{i=1}^n X_i, t = 0, 1, 2, \cdots$.

4. (30 points) Suppose that $X_1, \dots, X_n$ are random samples from a N(μ, 1) distribution.

   (a) Derive the MLE of μ (needs to check that it is indeed an MLE), then obtain (by using known results) the MLE of μ^2.

   (b) Find the uniformly minimum variance unbiased estimator of μ^2. State carefully what result(s) you are using.

   (c) Derive a size α likelihood ratio test for the null hypothesis $H_0 : \mu^2 = 0$ against the alternative hypothesis $H_1 : \mu^2 > 0$. (Hint: convert this to a test for μ.)

(d) Obtain the power function of the test in (c), with $\alpha = 0.05$, in terms of μ and $\Phi(\cdot)$ (the cdf of the $N(0, 1)$ distribution).

(e) Now consider testing $H_0 : \mu \leq 1$ against $H_1 : \mu > 1$. Explain why the test with rejection region of the form $\overline{X} > c$ is a UMP test. Specify the value of c if the size of the test is 0.05 for $n = 100$.

(f) Referring to part (e), find the type II probability at $\mu = 1.2$.

5. (20 points) Suppose you have a random sample of size n from a Uniform$(\theta,\ \theta + 1)$ distribution, $\theta \geq 0$. Let $X_{(1)}$ and $X_{(n)}$ denote the minimum and maximum sample value respectively. To test the null hypothesis $\theta = 0$ versus the one-sided alternative $\theta > 0$, use the test that rejects the null hypothesis if $X_{(1)} \geq c$ or $X_{(n)} \geq 1$. It is known that the joint pdf of the extremes $Y_1 = X_{(1)}$ and $Y_n = X_{(n)}$ is given by

$$g(y_1, y_n) = n(n-1)[F(y_n) - F(y_1)]^{n-2} f(y_1) f(y_n),$$

for $0 \leq \theta < y_1 < y_n \leq \theta + 1$, and is zero otherwise, where $F(\cdot)$ and $f(\cdot)$ are, respectively, the cdf and pdf of the said uniform distribution.

(a) Write the joint pdf $g(y_1, y_n)$ explicitly in terms of y_1, y_n, n and θ (i.e., no F and no f), and draw a diagram showing the domain where $g(y_1, y_n) \neq 0$.

(b) Determine c such that the test will have size α.

(c) Find the power function of the test.

(d) Let $\alpha = .10$. Find the necessary sample size so that the test will have power at least .80 if $\theta \geq 1$.