BASIC EXAM – COMPLEX ANALYSIS

AUGUST 2006

Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

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1. Evaluate

$$\int_C (2z-1)e^{z/(z-1)}dz$$

where $C = \{z \mid |z| = 2\}$ is the circle of radius 2 centered at the origin traversed once counterclockwise.

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2. Suppose γ is the square with vertices $\pm 2 \pm 2i$ traversed once counterclockwise. Compute

$$\int_\gamma \frac{z}{z^3+1}dz.$$

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3. Making sure to justify all estimates, use contour integration to evaluate

$$\int_0^\infty \frac{1+x^2}{1+x^4}dx.$$

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4. (a) For real constants $a < b$, determine the Laurent series expansion of

$$f(z) = \frac{1}{z-a} - \frac{1}{z-b}$$

convergent in the annulus $a < |z| < b$.

(b) Show that the function in part (a) has a primitive in the region

$$\Omega = \mathbb{C} \setminus \{x \in \mathbb{R} \mid a \le x \le b\}.$$

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5. Prove the following extension of the Argument Principle. Suppose g is holomorphic in an open connected set Ω. Let C be a counterclockwise simple closed curve lying in Ω whose interior is also contained in Ω. Let f be meromorphic in Ω, without zeros or poles on C itself and having zeros $z_1, \ldots, z_k$ and poles $p_1, \ldots, p_\ell$ in the interior of C, each repeated according to multiplicity. Prove that

$$\frac{1}{2\pi i}\int_C g(z)\frac{f'(z)}{f(z)}dz = \sum_{m=1}^{k} g(z_m) - \sum_{n=1}^{\ell} g(p_n).$$

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6. Consider the open horizontal strip

$$S = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi\}.$$

Write down explicitly all those conformal maps from S onto the open unit disc $\mathbb{D}$ which send $i\pi/2$ to the origin.

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7. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$. Show that

$$g(z) := \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

defines an entire function, and that for each fixed $0 < r < R$, there is a constant M such that $|g(z)| \leq M e^{|z|/r}$ for all $z \in \mathbb{C}$.

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8. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function that maps the open unit disc, $\mathbb{D}$, into itself and vanishes at the origin. Prove that

$$|f(z) + f(-z)| \leq 2|z|^2 \text{ for all } z \in \mathbb{D}.$$

Hint: Consider $\frac{f(z)+f(-z)}{2z}$.

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9. Show that the function $g(z) = z+3+2e^z$ has precisely one zero in the left-half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$. Be sure to state any theorem you use and check to make sure all its hypotheses are satisfied.

Hint: Consider zeros inside semi-circles of variable radius.

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10. Let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z_1 + a_0$ be a **monic** polynomial with complex coefficients. Show that there exists z with $|z| = 1$ and $|f(z)| \geq 1$, i.e.

$$\max_{|z|=1} |f(z)| \geq 1.$$

Hint: Consider the point at infinity.
