DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Consider the circuit equation

$$LI'' + RI' + IC = 0$$

where $L, C > 0$ and $R \geq 0$.

(a) Rewrite the equation as a two-dimensional system.

(b) Show that the origin is asymptotically stable if $R > 0$ and neutrally stable if $R = 0$.

(c) Classify the fixed point at the origin, depending on whether $R^2C - 4L$ is positive, negative, or zero, and sketch the phase portrait in all three cases.

2. Consider the system $x' = y^3 - 4x$, $y' = y^3 - y - 3x$.

(a) Find all the fixed points and classify them.

(b) Show that $|x(t) - y(t)|$ approaches 0 as t approaches ∞ for all other trajectories. (Hint: Form a differential equation for $x - y$.)

(c) Draw the phase portrait.

3. In a certain fishery, assume that fish are caught at a constant rate h (harvesting rate) independent of the size of the fish population. K is

the natural capacity of the fishery, r is the natural growth rate. Then the number of fish in the fishery at any time t, $y(t)$, satisfies

$$\frac{dy}{dt} = r(1 - \frac{y}{K})y - h$$

(a) Determine a condition (an inequality between h, r, K) such that any initial fish population will eventually become depleted (that is, $y(t) = 0$ for some $t > 0$).

(b) On the other hand, under what conditions is there a stable fixed point y^*? Give an explicit formula for y^*.

4. Consider the initial boundary value problem for a function $u(x,t)$:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$
$$u(L,t) = B, \quad \frac{\partial u}{\partial x}(0,t) = Q$$
$$u(x,0) = u_0(x).$$

(a) Explain in physical terms the meaning of the constants D, B, Q when u represents the temperature in a rod over the interval $0 \leq x \leq L$.

(b) Determine the equilibrium solution $u^*(x)$ that is independent of time.

(c) The general solution with initial solution $u_0(x)$ has the form

$$u(x,t) = u^*(x) + \sum_{k=1}^{\infty} e^{-\lambda_k t}\phi_k(x)\,.$$

Exhibit both the differential equation and the boundary conditions that each function ϕ_k must satisfy.

5. Consider the Laplace equation

$$\triangle u = u_{xx} + u_{yy} = 0 \text{ in } x^2 + y^2 < R^2$$

in a disk of radius R. Find the solution u satisfying the boundary condition

$$u(R, \theta) = 3\cos(2\theta) + 5\sin(\theta) \qquad (\ \theta \in [0, 2\pi]\ )$$

6. The motion of a string with friction is modeled by the modified wave equation

$$u_{tt} - c^2 u_{xx} + \gamma u_t = 0.$$

Here $\gamma > 0$ and $u_x(0, t) = u_x(L, t) = 0$.

(a) Let

$$E = \frac{1}{2}\int_0^L ({u_t}^2 + c^2 {u_x}^2) dx$$

and derive the identity

$$\frac{\partial E}{\partial t} = -\gamma \int_0^L {u_t}^2 dx$$

(b) Interpret this identity in terms of dissipation of energy.

7. Consider the following hat function $f(x)$ given by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \pi/2 \\ \pi - x & \text{if } \pi/2 \leq x \leq \pi \end{cases}$$

(a) Find the Fourier sine series for $f(x)$.

(b) Find the Fourier sine series for $f'(x)$.

(c) What can you say about their convergence at $\pi/2$.