DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

AUGUST 2006 ?????

Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the THREE parts.

Part I.

1. For any prime p, show that a finite Abelian p-group is generated by its elements of highest order.

2. (a) Show that if H is a normal subgroup of the finite group G, and if a prime p does not divide $[G : H]$, then H contains every Sylow p-subgroup of G.
(b) Give a counterexample when H is not normal. Justify your answer.

3. Let p be a prime. For any integer $n > 0$, denote by $\mathbf{F}_{p^n}$ the finite field with p^n elements. The Frobenius automorphism

$$\mathrm{Fr}_{p^n} : \mathbf{F}_{p^n} \rightarrow \mathbf{F}_{p^n}$$

is a bijective map from $\mathbf{F}_{p^n}$ to itself, and hence can be viewed as an element of the symmetric group S_{p^n}. In particular, it makes sense to ask if Fr_{p^n} belongs to the alternating group A_{p^n} or not.
(a) If n is odd, show that $\mathrm{Fr}_{p^n} \in A_{p^n}$. (**Hint:** consider the orbits of the Fr_{p^n}-action on $\mathbf{F}_{p^n}$)
(b) If the prime p is odd, show that $\mathrm{Fr}_{p^2} \in A_{p^2}$ if and only if $p \equiv 1 \pmod 4$.
(c) If the prime p is odd, determine whether or not Fr_{p^4} is in A_{p^4}.

Part II.

1. Show that an integral domain is a UFD if and only if *both* of the following conditions hold:

- every ascending chain of principal ideals terminates, and
- every irreducible element is prime.

2. Consider the following matrix over a field F:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Turn the F-vector space $M := V^6$ into a finitely generated $F[x]$-module in the usual way: for any $f(x) \in F[x]$ and any $v \in M$, set $f \cdot v$ to be the matrix $f(A)$ times the column vector v. With respect to this action, $N := \ker A$ is a $F[x]$-submodule of M.
(a) Decompose M into a direct sum of simple $F[x]$-modules.
(b) Do the same for $M \otimes_{F[x]} N$.

3. Determine the Jordan form of all *nilpotent* 6×6 matrices over the finite field $\mathbf{F}_3$. Justify your answer.

Part III.

1. Let L/K be a finite Galois extension. For any prime divisor p of $[L : K]$, show that there exists a subfield F of L such that $[L : F] = p$ and $L = F(\alpha)$ for some $\alpha \in L$.

2. Consider the extension $F = \mathbf{C}(t^4)$ over $L = \mathbf{C}(t)$, where t is a variable.
   (a) Show that L is the splitting field of $x^4 - t^4$ over F.
   (b) Show that $x^4 - t^4$ is irreducible over F.
   (c) Determine $\mathrm{Gal}(L/F)$.

3. (a) Let $q > 2$ be a prime. For any distinct integers u, v, show that at least one of u, v or uv is a square modulo q.
   (b) Let $f(x) \in \mathbf{Z}[x]$ be a quartic polynomial with Galois group $\mathbf{Z}/2 \times \mathbf{Z}/2$. Show that $f(x)$ is *reducible* modulo every prime > 3.