Department of Mathematics and Statistics
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Basic Exam: Topology

September 2, 2005

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Consider these topologies on $\mathbb{R}$:
   (a) discrete (= fine topology),
   (b) standard(= order topology), and
   (c) Zariski (= finite complement topology).
   For each of these topologies, determine whether or not the interval $(-\infty, 2005)$ is:
      i. open,
      ii. closed,
      iii. compact (in the subspace topology).

2. Let $\mathbb{R}$ be the real line with usual topology and $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $\mathbb{R}^\omega = \Pi_{i\in\mathbb{N}}\mathbb{R}$, with the product topology. Let K be a compact subset of $\mathbb{R}^\omega$. Prove that K has empty interior (the interior of K is the union of all subsets U of K that are open in $\mathbb{R}^\omega$).

3. Let X be a compact metric space and $f : X \to \mathbb{R}$ be continuous.
   (a) Define what it means for f to be uniformly continuous.
   (b) Prove that f is uniformly continuous.

4. Let $\mathcal{C}$ be the set of continuous functions from $[0, 1]$ to $\mathbb{R}$. Let $f \in \mathcal{C}$ satisfy $f(0) = 0 = f(1)$, but $f(x)$ is not zero for all x. Does the sequence of functions $f_n \in \mathcal{C}$, defined by $f_n(x) = f(x^n)$ for $x \in [0, 1]$; converge in
   (a) topology $\mathcal{T}_p$ of pointwise convergence on $\mathcal{C}$?
   (b) compact-open topology $\mathcal{T}_c$ on $\mathcal{C}$?
   (c) uniform topology $\mathcal{T}_u$ on $\mathcal{C}$?

5. Let X be compact, Y be Hausdorff and $f\colon X \to Y$ be a continuous bijection.

   (a) Prove that f is a homeomorphism.

   (b) Show with an example, that if X is not compact, then the statement does not hold.

6. Consider the quotient space $X = \mathbb{R}/\mathbb{Q}$. Prove the following statements:

   (a) X is compact.

   (b) X is *not* Hausdorff.

   (c) X is path connected.

7. Let $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$ be a sequence of nested closed connected sets in a Hausdorff space X.

   (a) Prove that if C_1 is compact, then $\cap_{n=1}^{\infty} C_n$ is connected and compact in the subspace topology.

   (b) Show that the compactness condition on C_1 is necessary in order to conclude that $\cap_{n=1}^{\infty} C_n$ is connected. (Hint: Find an example in $X = \mathbb{R}^2$.)