DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
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Work all problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (24 pts) Let $X_1, \ldots, X_n$ denote a random sample from the exponential distribution, with unknown location parameter θ and scale parameter λ , whose density is given by
$$p(x|\theta, \lambda) = \lambda \ \exp[-\lambda(x-\theta)], \ \ \theta \le x < \infty,$$
where $-\infty < \theta < \infty$ and $0 < \lambda < \infty$.
   (a) Find the mean and the variance of $X - \theta$.
   (b) Find the mean (μ) and the variance (σ^2) of X.
   (c) Find the MLEs of θ and λ. (NOTE: You must justify why your answers are MLEs.)
   (d) Give the MLEs of μ and σ^2, with brief explanations.

2. (36 pts) Suppose $X_1, \ldots, X_n$ is a random sample from the distribution with density
$$f(x;\theta) = \ \theta x^{\theta-1}, \quad \text{where } 0 < x < 1, \ \ \theta > 0$$
and let $T = -(1/n)\sum_{i=1}^{n} ln(X_i)$. It is known that $E(T) = 1/\theta$, $\text{Var}(T) = 1/(n\theta^2)$.
   (a) Compute $I_1(\theta)$, the Fisher information in a single observation.
   (b) Find the Cramér-Rao lower bound for unbiased estimators for $1/\theta$.
   (c) Explain how one can conclude that T is a UMVU estimator for $\rho = 1/\theta$.
   (d) Knowing that the mean of X is $\mu = \theta/(1+\theta)$, first justify that $\hat{\mu} = 1/(T+1)$ is a consistent estimator of μ; then find the asymptotic distribution of $\hat{\mu}$.
   (e) From Bayesian point of view, the pdf given above can be regarded as a conditional pdf given $\Theta = \theta$. Assume that Θ has a prior pdf $g(\theta) = \exp(-\theta), \ \ \theta > 0$.
      i. Find the posterior pdf of Θ given $X_1, \ldots, X_n$.
      ii. Find the Bayes estimate of θ under squared error loss function. [Hint: Associate the posterior pdf with a Gamma distribution which has pdf of the form
$$h(y|\alpha, \beta) = [\Gamma(\alpha)\beta^{\alpha}]^{-1} y^{\alpha-1} e^{-y/\beta}, \ \ y > 0.$$
]

3. (16 pts) A measuring instrument is run n times on a known standard which has a known value μ_0. The resulting observations $X_1, \ldots, X_n$ are assumed to be independent and normally distributed with mean μ_0 and an unknown variance σ^2. We wish to test that the instrument is sufficiently precise by testing the hypothesis $H_0: \ \sigma \leq 2$ versus $H_1: \ \sigma > 2$.

   (a) Show that the uniformly most powerful test of level α is to reject H_0 if and only if

   $$\sum_{i=1}^{n} (X_i - \mu_0)^2 > c.$$

   It is okay to appeal to a general result, but state clearly and completely what result you are appealing to. State exactly how c is obtained in general.

   (b) Exhibit explicitly the power function for the test in (a) and show that the power function is an increasing function of the true value σ.

4. (24 pts) Let $X_1, \ldots, X_n$ be iid, with pdf

   $$f_\alpha(x) = \frac{1 + \alpha x}{2}, \quad -1 \leq x \leq 1,$$

   where the parameter α satisfies $-1 \leq \alpha \leq 1$. Suppose that we observe only $Y_i = I_{[X_i > 0]}$, rather than X_i itself.

   (a) Describe the distributions of $Y_1, \ldots, Y_n$, then find the MLE $\hat{\alpha}$ of α based on $Y_1, \ldots, Y_n$. [Hint: First, find the MLE of the expected value of Y_i.]

   (b) Determine whether $\hat{\alpha}$ is consistent for α. [You may use known theorems, but point out which theorems that you appeal to.]

   (c) Display an approximate 95% CI (confidence interval) for α, and explain under what conditions such a CI is valid.

   (d) Suppose that, based on a sample of size 100, we obtained the interval (.25, .50) in part (c). Explain the meaning of the statement "(.25, .50) is a 95% CI for α".