DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – PROBABILITY
August 31, 2005

Work all problems. 60 points are needed to pass at the Master's level and 75 to pass at the Ph.D. level.

1. (25 points) Suppose that X and Y have joint density function $f(x, y) = c(x+y)I_{(0,1)}(x)I_{(0,1)}(y)$.
   (a) What must the constant c be in order for $f(x, y)$ to be a density function?
   (b) What are the mean and variance of X?
   (c) Find the conditional distribution of Y given $X = x$.

2. (15 points) Suppose the number of customers Y entering a bank in a one hour period is distributed Poisson with mean 10 (pmf: $e^{-10}10^y/y!$). Suppose that given $Y = y$ the total time T needed to service the y customers has an exponential distribution with mean $3y$ (pdf: $e^{-t/(3y)}/(3y)$). Find the unconditional mean and variance of T.

3. (15 points) Let X_1 and X_2 be iid with Uniform(-1,1) distributions.
   (a) Find the pdf of $Y = X_1^2$.
   (b) Let $Z = X_1X_2$. Are Y and Z independent? Why or why not?

4. (20 points) Two players, A and B, are playing a game. The game consists of a series of trials, and the first player to win two more trials than the other will win the game. Suppose that each trial is *iid*, and the probabilities that either player wins a trial are:

$$\begin{aligned} Pr(A \text{ wins a trial}) &= p \\ Pr(B \text{ wins a trial}) &= q = 1 - p. \end{aligned}$$

   (a) What is the probability that A wins in exactly 6 trials?
   (b) What is the probability that A wins in exactly 2n trials?
   (c) What is the probability that A wins?

5. (25 points) Let X_t be the number of fish that a particular fishing boat catches on trip t. Suppose that the number of fish caught is independent from trip to trip, the mean number of fish caught on any particular trip is 49, and the variance is also 49.
   (a) Over the course of the season of 100 trips, what is the approximate probability that the mean number of fish that are actually caught is no less than 48? (You may leave your answer as a formula.)
   (b) Let Y_t be the profit from trip t. Due to market forces suppose that $Y_t = \log(X_t)$. Is the mean of Y_t greater than, less than, or equal to $\log(49)$ or do you need more information? (and why?)
   (c) Next, make the additional assumption that the number of fish caught on a particular trip has a Poisson distribution. What is the probability that no fish are caught on at least one out of 250 trips? (You may leave your answer as a formula.)

(d) A slightly more sophisticated model posits that on overcast days the expected number of fish caught are iid with mean μ_o and standard deviation σ_o. On sunny days the number caught are iid with mean and standard deviation μ_s and σ_s. Suppose that the probability of an overcast day is p. What are the marginal mean and variance of the number of fish caught under this model?