BASIC EXAM: NUMERICS

Do five of the following problems. All problems carry equal weight.
Passing Level:
Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1a) Solve the linear system:

$$x_1 + 2x_2 + 4x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$
$$2x_1 + x_2 + 5x_3 = 9$$

using Gaussian elimination.

b) Find the LU decomposition of the matrix of the above system.

2. Consider the following 2×2 matrix

$$A = \begin{bmatrix} a & -b \\ -a & a \end{bmatrix},$$

where a and b are real numbers, satisfying $a > 0$, $b > 0$ and $a > b$.

a) Examine whether the Jacobi iteration is convergent for this class of matrices.

b) Examine whether the Gauss-Seidel iteration is convergent for this class of matrices.

3. Consider the equation $x^2 + x - 2 = 0$ and in particular its positive root.

a) Devise 3 different fixed point iteration schemes that could converge to the (positive) solution of the above equation.

b) Find intervals where these schemes would be guaranteed to converge (i.e., where your schemes satisfy relevant conditions for convergence) and show that the relevant conditions are indeed satisfied.

c) Write a very brief script implementing (in your favorite programming language) one of these fixed point schemes.

4a) Derive the numerical differentiation formula of the form

$$f'(x) \approx S_h f = af(x) + bf(x+h) + cf(x+2h)$$

with the highest possible degree of precision.

b) Bound the error in the approximation, assuming f is smooth enough.

c) Now try to combine S_h and $S_{h/2}$ to produce a more accurate approximation.

5. Assume that you use a set of $n+1$ points, (x_0, y_0), ..., (x_n, y_n) to approximate a function $f(x)$ passing through these points by an interpolating polynomial $p_n(x)$.

a) Prove that

$$f[x, x, \ldots, x] = \frac{f^{(n)}(x)}{n!}$$

where the left hand Newton divided difference has $n+1$ arguments all equal to x and in the right hand side the superscript denotes the n-th derivative. Hint: use (without proof) two separate expressions, one from Lagrange interpolation and one from Newton divided differences, for the error of the above mentioned polynomial approximation and set them equal.

b) Application: assume $f(0) = 1$ and $f(1) = 2$, as well as $f'(0) = 2$ and $f'(1) = 3$. Use Newton divided differences to construct the relevant interpolating polynomial of third order and write an expression for the error in the approximation.

6. Consider the function $f(x) = \sin(x)$ in the interval $[0, \pi/2]$ and consider the points $x_0 = 0$, $x_1 = \pi/4$ and $x_2 = \pi/2$ in this interval (recall that $\sin(\pi/4) = \sqrt{2}/2$).

a) Find the interpolating quadratic polynomial to $f(x) = \sin(x)$, using the value of the function only at x_0, x_1 and x_2.

b) Integrate the function to obtain an approximation to $\int_0^{\pi/2} \sin(x)dx$. Make sure that the answer is exactly the same as the one you expect from Simpson's rule.

c) Give (or derive) an expression for the error of your approximation. Compare the true value of the integral with the approximate one evaluated above to examine the validity of the relevant error estimate.

7. Consider the initial value problem problem $y' = f(t, y)$, $y(t^0) = y^0$ and the multistep formula

$$y_{n+1} = (1-a)y_n + ay_{n-1} + \frac{h}{12}\left((5-a)f_{n+1} + 8(1+a)f_n + (5a-1)f_{n-1}\right).$$

Here a is a real parameter, $f_n = f(t_n, y_n)$, $t_n = t_0 + nh$, and y_n is an approximation of $y(t_n)$.

a) Show that the above method is of order 3 for *general* a.

b) Find a *particular* value of a such that the order of the method is 4.

c) State what it means for a multistep method to be A-stable for the initial value problem. Is the method of part (b) A-stable?