University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
August 2005

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

(1) (a) Prove that a compact Riemannian manifold is geodesically complete.
(b) Give a counterexample to the converse of the previous statement.
(c) Give an example showing that a noncompact Riemannian manifold need not be geodesically complete.

(2) Define two vector fields in $\mathbf{R}^3 - \{x = 0 \cup y = 0\}$ by

$$X = \partial_x - \frac{z}{x}\partial_z, \quad Y = \partial_y - \frac{z}{y}\partial_z.$$

(a) Show that this pair of vector fields gives an involutive distribution in the positive orthant

$$\mathbf{R}^3_+ = \{(x,y,z) \mid x,y,z > 0\}.$$

(b) Describe the integral submanifold through any point in $\mathbf{R}^3_+$.
(c) Sketch the integral submanifold through $(1,1,1)$.

(3) Let $U \subset GL_3(\mathbf{R})$ be the subgroup of upper-triangular matrices with determinant one. For coordinates on U, use the restriction of the standard coordinate functions x_{ij} on $GL_3(\mathbf{R})$.
(a) Show that U is a Lie subgroup and describe its Lie algebra.
(b) Explicitly compute a basis of left-invariant vector fields on U in terms of the basic frame $\{\partial_{x_{ij}}\}$.
(c) Explicitly compute a basis of left-invariant 1-forms on U in terms of the basic coframe $\{dx_{ij}\}$.

(4) Let X be the subset of $\mathbf{P}^{m+n+1}(\mathbf{R})$ given by the zero set of the polynomial $x_0^2 + \cdots + x_m^2 - y_0^2 - \cdots - y_n^2$.
(a) Show that X is a manifold diffeomorphic to $S^m \times S^n$.
(b) Compute the De Rham cohomology groups $H^*(X)$.

(5) Let M be a manifold with De Rham cohomology groups $H^*(M)$. Suppose $\alpha \in H^p(M)$ is represented by the differential form η, $\beta \in H^q(M)$ is represented by the form θ, and $\gamma \in H^r(M)$ is represented by the form ψ. Suppose also that there exist differential forms ω_1, ω_2 such that $\eta \wedge \theta = d\omega_1$ and $\theta \wedge \psi = d\omega_2$.
(a) Show that the differential form $\zeta := \eta \wedge \omega_2 - (-1)^p \omega_1 \wedge \psi$ is closed.

(b) Show that the cohomology class of ζ depends only on the classes of α, β, γ, and not on the forms η, θ, ψ representing these classes.

(6) Let M be a manifold.
  (a) Define what it means for a vector bundle $E \to M$ to be *trivial*.
  (b) What is the relationship between triviality of vector bundles and parallelizability of a manifold M?
  (c) Show that if E, E' are two trivial vector bundles over M, then $E \oplus E'$ is trivial.
  (d) If $E \to M$ is a vector bundle and $E \oplus E$ is trivial, is E trivial? What if $E \otimes E$ is trivial?

(7) Let $X \subset \mathbf{R}^3$ be the torus with parameterization

$$((a+b\cos u)\cos v, (a+b\cos u)\sin v, b\sin u), \quad 0 < b < a, \quad (u,v) \in \mathbf{R}^2.$$

Give X the induced metric.
  (a) Compute $*du, *dv$, and $*(du \wedge dv)$.
  (b) Compute a local expression in the coordinates (u,v) for the Laplacian operator Δ on functions and 2-forms.

(8) Consider the sequence of surfaces

$$S_n = \{(x,y,z) \mid z = (x^2+y^2)^n\}, \quad n \geq 1.$$

  (a) Write a parameterzation of S_n.
  (b) Compute the Gaussian curvature of S_n in terms of the coordinates you chose in part (a).
  (c) All the S_n pass through the point $(1,0,1)$. What happens to the sequence of Gaussian curvatures at this point as $n \to \infty$? Explain why this is geometrically plausible.