DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

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Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the four parts.

Part I.

1. Let G be a finite group acting transitively on a set S. Let $H \trianglelefteq G$ be a normal subgroup of G. Let $\mathcal{O}_1, \cdots, \mathcal{O}_s$ be the orbits of H acting on S.

(a) Prove that these orbits all have the same cardinality.
(b) Let $a \in S$ and let $G_a = \{g \in G \mid g.a = a\}$. Show that $s = |G : HG_a|$.

2: Let G be a finite group and P a non-trivial p-Sylow subgroup of G. Let H be the normalizer of P in G. Show that the normalizer of H in G is equal to H.

Part II.

All rings are commutative with 1, and all ring homomorphisms take 1 to 1.

3. Let k be a field. Recall that a k-algebra is a ring R with a multiplicative identity 1_R and a map of rings $k \to R$ that takes the multiplicative identity of k to 1_R.

(a) Let V and W be k-algebras. Show that $V \otimes_k W$ has a natural structure as a ring.
(b) Show that the rings $\mathbf{C} \otimes_{\mathbb{R}} \mathbf{C}$ and $\mathbf{C} \oplus \mathbf{C}$ are isomorphic rings.

4. Let R be a commutative ring with identity. Recall that an element $p \in R$ is prime if the ideal generated by p is a non-zero prime ideal.

Show that if an element in an integral domain is expressible as a product of primes, then that expression is unique up to multiplication by units and permutations of the elements in the expression.

Part III.

5. Let k be a field and V a vector space of finite dimension n over k. Let A be a linear transformation of V with minimal polynomial $(x-a)^n$ for some $a \in k$.

(a) Find the Jordan canonical form of A.
(b) Describe the set of linear maps from V to itself which commute with A.

6. Let M be a free $\mathbf{Z}$-module with basis $\{e_1, e_2, e_3\}$. Let N be the submodule of M generated by

$$\{e_1 - e_2 - e_3,\ e_1 - e_2 + e_3,\ -2e_1 + 10e_2 - 6e_3\}.$$

(a) Describe the isomorphism type of N as a $\mathbf{Z}$-module.
(b) Describe the isomorphism type of M/N as a $\mathbf{Z}$-module.
(c) Find a submodule N' of M such that $M = N + N'$ and the sum is a direct sum, or explain why no such N' exists.

Part IV.

7. Let $f(x) = x^n - 1$.

(a) Prove that the Galois group of $f(x)$ over the field of rational numbers is an abelian group.

(b) Find the smallest n such that the Galois group is not cyclic.

8. Let K be a finite, separable extension of the field k. Prove that if K is a splitting field over k then every irreducible polynomial in $k[x]$ that has a root in K splits completely in $K[x]$.