Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

Friday September 3, 2004

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Let $f(x) = e^x \cos x$, a function $\mathbb{R} \to \mathbb{R}$.
  (a) If both the source and target $\mathbb{R}$'s are given the finite complement topology, is f continuous?
  (b) If both the source and target $\mathbb{R}$'s are given the usual topology, show that f is neither an open nor a closed map.

(2) Let X be a compact, Hausdorff space. Let A, B be disjoint closed subsets of X. Show that there exist disjoint open subsets U and V containing A and B, respectively. (Hint: first start with the case when $B = \{b\}$ is a single point)

(3) Give the space $X = \mathbb{R}^\omega$ of sequences in $\mathbb{R}$ the product topology. Show that a compact subset $C \subset \mathbb{R}^\omega$ must have empty interior.

(4) Let X be a metric space.
  (a) Show that X has a countable dense subset if and only if X has a countable base for its topology.
  (b) Suppose that X is compact. Show that both conditions from part (a) hold (you only need to show one!).

(5) Let $f\colon X \to Y$ be a quotient map between spaces X and Y. Suppose that Y is connected, and that $f^{-1}(y)$ is connected for all $y \in Y$. Show that X is connected.

(6) Let X be the space of all functions $[0, 1] \to [0, 1]$, endowed with the uniform metric:

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Let $Y \subset X$ be the subspace of continuous functions. Show that Y is closed in X.

(7) Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$. Define an equivalence relation $\sim$ on X by saying $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = 2^n x_2$, $y_1 = 2^n y_2$ for some $n \in \mathbb{Z}$. Show that the quotient space $X/\sim$ is homeomorphic to $S^1 \times S^1$.

[hint: look at the annulus between circles of radius 1 and 2 with centers at $(0, 0)$]