DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
September 2004

Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Show that the solution to $\dot{x} = x^{1/3}$ starting with $x(0) = 0$ is not unique by finding two different solutions.

2. Consider the system

$$\frac{dx}{dt} = 1 - y$$
$$\frac{dy}{dt} = x^2 - y^2.$$

   (a) Determine all critical points of the system.

   (b) Find the corresponding linear system near each critical point.

   (c) Discuss the stability of the solution near each critical point.

3. For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N: this is called the "Allee effect". For example, imagine that it is too hard to find mates when N is very small and there is too much competition when N is large. Consider the population model

$$\dot{N}/N = r - a\,(N - b)^2$$

where r, a and b are positive parameters.

   (a) Find all the fixed points of the system, and find the value of N for which the effective growth rate $\dot{N}/N$ is maximal.

(b) Describe the conditions on r, a and b under which the population has a unique positive equilibrium.

(c) Under these conditions, sketch the solutions $N(t)$ for different initial values.

4. Solve the linear equation

$$(1 + x^2)u_x + u_y = 0$$

and sketch some of the characteristic curves.

5. Find the general solution of the boundary value problem $u_t = ku_{xx}$, $k > 0$, with $u(0, t) = 0$ and $u_x(l, t) = 0$ in the domain $[0, l]$. *Do not forget to show that no negative or zero eigenvalues are possible.*

6. Show for the equation $u_{tt} = c^2 u_{xx}$ that the energy

$$E = \frac{1}{2} \int_0^l u_t^2 + c^2 u_x^2 dx \tag{1}$$

is conserved. Show this *separately* for Dirichlet, Neumann and Robin boundary conditions in the domain $[0, l]$.

7. Solve $u_{xx} + u_{yy} = 0$ in the disk $r < a$ with the boundary condition $u = (\sin(\theta))^5$. Develop the solution starting from the separation of variables in polar coordinates, keeping in mind that

$$(\sin(\theta))^5 = (1/16)\sin(5\theta) - (5/16)\sin(3\theta) + 5/8\sin(\theta). \tag{2}$$