Department of Mathematics and Statistics
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Basic Exam: Advanced Calculus/Linear Algebra

August 30, 2004

Do 7 out of the following 9 problems. Indicate clearly which problems should be graded.

Passing standard: To pass at the Master's level it is sufficient to have 60% with **three** problems essentially complete (including at least one from each part). To pass at the Ph.D. level, 75% with **two questions from each part** essentially complete.

Part I
Linear algebra

Problem 1. Let $u = (u_1, u_2, u_3, u_4) = (5, -2, 3, 1)$ and $v = (v_1, v_2, v_3, v_4) = (-2, 1, -1, 2)$. We define a 4×4 matrix A by:

$$A = (a_{ij}) , \quad \text{where} \quad a_{ij} = u_i v_j ; \quad 1 \le i, j \le 4 .$$

1. Compute rank(A).
2. Compute Ker(A).
3. Compute Im(A).

Problem 2. Let V be a real vector space and $T\colon V \to V$ a linear transformation such that $T \circ T = I$, where I is the identity transformation. Show that

$$V_1 := \{v + T(v) : v \in V\} \text{ and } V_2 := \{v - T(v) : v \in V\}$$

are subspaces of V and $V = V_1 \oplus V_2$.

Problem 3. Let A be an $n \times n$ matrix with complex entries. Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalues of A counted with multiplicity. Show that

1. $\det(A) = \lambda_1 \cdots \lambda_n$
2. $\text{tr}(A) = \lambda_1 + \cdots + \lambda_n$

Problem 4. Let V be an n-dimensional vector space and $T\colon V \to V$ a linear transformation Suppose that $v_1, \ldots, v_n \in V$ are eigenvectors of T corresponding to distinct eigenvalues. Show that $v_1, \ldots, v_n$ are a basis of V.

Part II
Advanced Calculus

Problem 1. Find the points in the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \ : \ xy^2z^4 = 64\}$$

that are closest to the origin.

Problem 2. Let $\vec{F}$ be the vector field in $\mathbb{R}^3$

$$\vec{F}(x, y, z) = \frac{1}{x^2 + y^2 + z^2}\,(x, y, z)$$

Compute the flux:

$$\int_{S_a} \vec{F} \cdot dS\,,$$

where S_a is the sphere centered at the origin of radius a.

Problem 3. Compute the line integral:

$$\int_C 2xydx + x^2dy\,,$$

where C is the short piece of the ellipse

$$(x-2)^2 + 4y^2 = 4$$

going from the origin to the point $(2, 1)$.

Problem 4. Let $a_0 = 1/2$ and consider the sequence defined recursively by:

$$a_n = \ln(1 + a_{n-1})\,.$$

1. Show that $\lim_{n\to\infty} a_n = 0$.
2. Determine the radius of convergence of the power series:

$$\sum_{n=0}^{\infty} a_n x^n$$

Problem 5. Compute the iterated integral

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin x^2 \, dx dy$$