University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
September 1, 2004

Do 5 out of the following 7 questions. Indicate clearly which questions you want to have graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(u, v) = (\sinh u \cos v, \sinh u \sin v, v) .$$

a) Show that $M = f(\mathbb{R}^2) \subset \mathbb{R}^3$ is a 2-dimensional submanifold.
b) Compute the Gaussian curvature of M with the metric induced from $\mathbb{R}^3$.
c) Write the geodesic equations for M and determine if, suitably parametrized, any of the coordinate curves $\{u = \text{constant}\}$ or $\{v = \text{constant}\}$ are geodesics on M.
d) Draw a picture of the surface M.

Problem 2. A $2n$-dimensional manifold (M, g) is said to be *symplectic* if there exists a closed 2-form ω on M such that

$$\omega^n \ := \ \overbrace{\omega \wedge \cdots \wedge \omega}^{n \text{ times}}$$

is nowhere zero. Determine which of the following 4-manifolds are symplectic. Justify your answers.

a) $\mathbb{R}^4$.
b) S^4.
c) $S^2 \times S^2$.

Problem 3. Consider the vector fields $V = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$ and $W = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$.

a) Determine the open set $U \subset \mathbb{R}^3$ over which V and W span a 2-dimensional distribution, i.e., a rank 2 subbundle $E \subset TU$ of the tangent bundle TU.
b) Find a 1-form α in U such that

$$E(p) \ = \ \{X \in T_p(U) \ : \ \alpha(p)(X) = 0\} \subset T_p(U) \ ; \ \text{for all } p \in U.$$

c) Show that E is integrable.
d) Find the integral submanifolds of E.

Problem 4. Let X be the C^∞ vector field on $\mathbb{R}^{n+1}\backslash\{0\}$:

$$X = \sum_{i=0}^{n+1} x_i \frac{\partial}{\partial x_i}.$$

a) Prove that X is a complete vector field and compute the one-parameter group of diffeomorphisms (flow) of X.

b) Let $\pi\colon \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ denote the natural projection. Show that for every $\alpha \in \Lambda^k(\mathbb{P}^n)$, $1 \le k \le n$,

$$\iota_X(\pi^*(\alpha)) = 0.$$

Problem 5. Let $\mathbb{R}^3$ be endowed with the Heisenberg product:

$$(x', y', z') * (x, y, z) := (x' + x, y' + y, z' + z + x'y)$$

You may assume as given that $(\mathbb{R}^3, *)$ is a Lie group.

a) Find a basis of left-invariant vector fields for $(R^3, *)$. Express your answer in terms of the coordinate frame $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$.

b) Find a basis of left-invariant 1-forms on $(R^3, *)$. Express your answer in terms of the coordinate coframe $\{dx, dy, dz\}$.

c) Find a left-invariant metric on $(R^3, *)$. Express your answer in terms of the coordinate coframe $\{dx, dy, dz\}$.

d) Let $\{\omega^i_j\,;\ 1 \le i, j \le 3\}$ denote the connection forms of the Riemannian (Levi-Civita) connection of the metric constructed in part **c)** relative to the left-invariant frame constructed in part **a)**. Show that

$$\omega^i_j + \omega^j_i = 0.$$

Problem 6. Prove or disprove the following statements:

a) Let $\alpha \in \Lambda^1(S^2)$ and suppose $T^*(\alpha) = \alpha$ for all $T \in SO(3)$. Then $\alpha = 0$.

b) If n is odd then every n-form $\alpha \in \Lambda^n(\mathbb{P}^n)$ vanishes at some point.

c) If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a submersion, then $M = f^{-1}(0)$ is an orientable manifold.

Problem 7. Let $E \to M$ be a real vector bundle of rank r with connection ∇. Show that the following statements are equivalent:

a) ∇ is flat, i.e., its curvature $R^\nabla = 0$.

b) Near each point there is an open neighborhood $U \subset M$ and a local framing $(\psi_1, \ldots, \psi_r)$ of E over U such that $\nabla\psi_k = 0$, i.e., all the local sections ψ_k of the frame are parallel.