Department of Mathematics and Statistics
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**ADVANCED EXAM — DIFFERENTIAL EQUATIONS**
**AUGUST 30, 2004**

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. Assume $\Omega$ is a bounded, connected open subset of $\mathbb{R}^n$ with smooth boundary. Show that for $1 \leq p \leq \infty$, there is a constant $C = C(n, p, \Omega)$ such that

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C\| \bigtriangledown u\|_{L^p(\Omega)}$$

where

$$(u)_\Omega = \frac{1}{|\Omega|} \int_\Omega u(x)dx$$

**HINT:** Prove first the inequality for smooth functions.

2. Let $u = u(x,t), x \in \mathbb{R}^n, t > 0$ be a smooth solution to the wave equation

$$(2) \quad u_{tt} - \Delta u = 0 \qquad x \in \mathbb{R}^n, \ \ t \in (0, T)$$

a) Show that the quantity

$$e(t) = \frac{1}{2} \int_{B(x_0, T-t)} [u_t^2 + | \bigtriangledown u|^2] dx$$

where $x_0 \in \mathbb{R}^n$ is an arbitrary point in $\mathbb{R}^n$ and $B(x_0, r)$ denotes a ball with center at $x_0$ and radius $r$, satisfies

$$\frac{d}{dt} e(t) \leq 0 \qquad t \in (0, T)$$

b) Use (a) to state and prove a uniqueness result for a suitable initial value problem for equation (2).

3. a) Study the linear stability of all steady states of the system

$$(1) \begin{cases} w' = av - w - b \\ v' = v(\frac{1}{2} - v)(v - 1) - w \end{cases} , \quad a > 0$$

for all possible choices of constants $a > 0, \ b \in \mathbb{R}$.

b) Prove that for a suitable parameter regime ($a > 0, \ b \in \mathbb{R}$), (1) has at least one nontrivial periodic solution.

4. a) Consider the functional

$$E[u] = \int_\Omega \frac{1}{2} | \bigtriangledown u|^2 - uf dx,$$

($\Omega \subset \mathbb{R}^n$ bounded, $\partial\Omega$ smooth) where $f$ is a given $L^2(\Omega)$ function. Show that $E[u]$ is bounded from below over all $u \in H_0^1(\Omega)$. By employing a minimizing sequence, show that $E[u]$ has a minimizer in $H_0^1(\Omega)$.

b) Prove that the minimizer in (a) is unique in $H_0^1(\Omega)$.

5. Consider the system on $\mathbb{R}^3$ given by

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_1(\alpha - x_1) \end{aligned}$$

where $\alpha > 0$ is constant.

a) Show that there is no solution running from $(0,0,0)$ at $-\infty$ to $(\alpha, 0, 0)$ at $+\infty$.

**HINT:** Construct a suitable Liapunov function.

b) Show by linearization methods that for small $\alpha$, $0 < \alpha << 1$, there is exactly one solution (modulo translations) running from $(\alpha, 0, 0)$ at $t = -\infty$ to $(0,0,0)$ at $t = +\infty$.

6. Consider the system

$$\begin{aligned} x' &= x^2y^3 + 5xy^4 - x^9 \\ y' &= x^4y - 3x^3y^3 - y^7 \\ z' &= -(1+x^2)z + x^3y^3 \end{aligned}$$

Find positive numbers P,Q,R such that if $|x(0)| < P$, $|y(0)| < Q$, and $|Z(0)| < R$, then these inequalities also hold for the solution for all $t > 0$.