NAME:

Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and f and g complex-valued measurable functions on X. We define

$$\rho(f) \;=\; \int_X \frac{|f|}{1+|f|}\, d\mu$$

and set $d(f,g) = \rho(f-g)$.

(a) Prove that the space of complex-valued measurable functions is a metric space with metric $d(\cdot\,,\,\cdot)$. Two functions are identified if they are equal μ-a.e.

(b) Prove that $\lim\limits_{n\to\infty} d(f_n, f) = 0$ if and only if $f_n \to f$ in measure.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $1 < p < \infty$ and choose q so that $\frac{1}{p} + \frac{1}{q} = 1$. Consider the spaces $L^p(\mu)$ and $L^q(\mu)$ of real-valued measurable functions. For $g \in L^q(\mu)$ and $f \in L^p(\mu)$ we define

$$T_g(f) \;=\; \int_X f\, g\, d\mu\,.$$

(a) Prove that $T_g(f)$ is finite using a well known inequality.

(b) Prove that as an operator from $L^p(\mu)$ into $\mathbb{R}$, $\|T_g\| \;\leq\; \|g\|_q$.

(c) Prove that, in fact, $\|T_g\| \;=\; \|g\|_q$. *Hint:* Consider $f = \operatorname{sign}(g)\, |g|^{q-1}$.

3. Let H be a Hilbert space with scalar product $\langle \cdot\,,\,\cdot \rangle$ and norm $\|\cdot\|$.

(a) State the Schwarz inequality.

(b) Prove that for all $x, y \in H$

$$\|y\|^2 \left[\|x\|^2 \|y\|^2 - |\langle x\,,\,y \rangle|^2 \right] \;=\; \Big\| \|y\|^2 x - \langle x\,,\,y \rangle y \Big\|^2 \tag{1}$$

(c) Use part (b) to derive the Schwarz inequality and to determine when equality holds in the Schwarz inequality.

4. Let ℓ^∞ denote the space of real sequences $x = (x_1, x_2, \ldots)$ such that

$$\|x\|_\infty = \sup_{n \geq 1} |x_n| < \infty \, .$$

Define

$$C = \{x \in \ell^\infty \ : \ \lim_{n \to \infty} x_n = x_\infty \text{ exists}\} \, .$$

(a) Prove that C is a closed linear subspace of ℓ^∞.

(b) Prove that C is nowhere dense; i.e., the interior of C is empty.
Hint: Given $x = (x_1, x_2, \ldots) \in C$ and $\varepsilon > 0$, consider $y = (y_1, y_2, \ldots)$, where $y_n = x_n + \varepsilon(-1)^n$.

5. Let $f \in L^1(\mathbb{R})$ and let g be a bounded measurable function. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g(x)\ [(f(x+1/n) - f(x)]\ dm(x) \ = \ 0\,.$$

Hint: Approximate f by a suitable continuous function.

6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) State the Radon-Nikodym Theorem.

(b) Prove that the Radon-Nikodym derivative $d\nu/d\mu$ has the following properties. Part (ii) is known as the chain rule.

(i) Let ν be a finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$. If $f \in L^1(\nu)$, then

$$\int_X f \, d\nu = \int_X f \frac{d\nu}{d\mu} \, d\mu \, .$$

(ii) Let ν and λ be finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$. Then $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \; \lambda\text{-}a.e..$$

7. Let φ be a nonnegative measurable function on $[0, \infty)$ such that

$$g(r) \;=\; \int_0^\infty e^{rx}\,\varphi(x)\,dm(x)$$

is finite for all $r \geq 0$.

(a) Prove that the function $g(r)$ is continuous for $r \geq 0$ by showing that for any nonzero sequence $\{h_n\}_{n\geq 0}$ with $\lim_{n\to\infty} h_n = 0$

$$\lim_{n\to\infty} g(r+h_n) \;=\; g(r)\,.$$

(b) Prove that the function $g(r)$ is differentiable for $r > 0$ by showing that for any nonzero sequence $\{h_n\}_{n\geq 0}$ with $\lim_{n\to\infty} h_n = 0$ the limit

$$\lim_{n\to\infty} \frac{g(r+h_n)-g(r)}{h_n}$$

exists and does not depend on the sequence $\{h_n\}_{n\geq 0}$. What is the value of the derivative $g'(r)$? Justify all your steps.

Hint: You may use (without proof) that, for all $x \geq 0$ and all $a \in I\!R$, $a \neq 0$

$$\left|\frac{e^{ax}-1}{a}\right| \;\leq\; e^{(1+|a|)x}\,.$$

8. (a) Let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable sets on $[0,1]$ and let m be Lebesgue measure on $[0,1]$. For $A \in \mathcal{M}$ we define

$$\mu\{A\} = m\Big\{A \cap [0, \tfrac{2}{3}]\Big\} \text{ and } \nu\{A\} = m\Big\{A \cap [\tfrac{1}{3}, 1]\Big\}.$$

Determine the Lebesgue decomposition of ν with respect to μ.

(b) Let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable sets on $[0,1]$, let m be Lebesgue measure on $[0,1]$, and let $m \times m$ be the product measure on $[0,1] \times [0,1]$. Define the function

$$g(x,y) \;=\; \begin{cases} 2 & \text{for } \; 0 \le y \le x \le 1 \\ 0 & \text{for } \; 0 \le x < y \le 1\,. \end{cases}$$

For $A \in \mathcal{M}$, define the measure

$$\tau(A) \;=\; \int_{A \times [0,1]} g \, d(m \times m)\,.$$

Prove that τ is absolutely continuous with respect to m and compute the Radon-Nikodym derivative $d\tau/dm$.
Hint: Use the Fubini-Tonelli theorem to write $\tau(A) = \int_0^1 1_A(x)\, f(x)\, dm(x)$ for an appropriate function $f(x)$.