DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

BASIC EXAM - NUMERICS

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Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. Find the polynomial of least order satisfying

$$p(a) = 2, \quad p(b) = 7, \quad p''(a) = 1 \quad \text{and} \quad p''(b) = 1.$$

2. We will attempt to find π as the root of the function

$$f(x) = 1 + \cos x.$$

   (a) Does Newton's method converge, and at what rate?

   (b) Find a method of the form

$$x_{k+1} = x_k - C\frac{f(x_k)}{f'(x_k)}$$

   which converges quadratically.

3. Derive the two point Gaussian quadrature formula for

$$I = \int_0^1 x f(x) dx$$

with weight function $w(x) = x$ and the error term.

4. We will investigate the stability of the midpoint scheme for solving the differential equation $\frac{dy}{dx} = f(x, y)$,

$$y_{i+1} = y_{i-1} + 2\,h\;f(x_i, y_i).$$

Suppose that $f(x, y) = \lambda y$, so we're solving the equation

$$\frac{dy}{dx} = y, \quad \text{and take} \quad y(0) = 1.$$

   (a) Write down the exact solution to the differential equation, and the midpoint scheme for approximating this solution.

   (b) Solve the difference equation, and determine whether or not the scheme is stable.

5. For the trapezoidal rule(denoted by I_n^T) for evaluating

$$I = \int_a^b f(x)dx,$$

we have the asymptotic error formula

$$I - I_n^T = -\frac{h^2}{12}[f'(b) - f'(a)] + O(h^4)$$

and for the midpoint formula I_n^M, we have

$$I - I_n^M = \frac{h^2}{24}[f'(b) - f'(a)] + O(h^4)$$

provided f is sufficiently differentiable on $[a, b]$. Using these results, obtain a new numerical integration formula $\bar{I}_n$ combining I_n^T and I_n^M, with a higher order of converegence. Write out the coefficients to the new formula $\bar{I}_n$.

6. Let

$$H = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

be the 3×3 Hilbert matrix.

a) Find the decomposition $H = L\,D\,L^T$, where D is diagonal and L is lower triangular with 1's on the diagonal.

b) Give an example of a 3×3 symmetric matrix M which **cannot** be decomposed in this way.

7. (a) Show that the family P $=\{1, sin(x), cos(x), ..., sin(nx), cos(nx)\}$ is orthogonal on $[0, 2\pi]$ with respect to

$$< g, h >= \int_0^{2\pi} g(x)h(x)dx.$$

Hint: use the trig identities

$$\sin(A + B) = \sin A\ \cos B + \cos A\ \sin B,$$
$$\cos(A + B) = \cos A\ \cos B - \sin A\ \sin B.$$

(b) Derive the continuous least squares approximation to $f(x)$ on $[0, 2\pi]$ using P.