University of Massachusetts
Department of Mathematics and Statistics
Advanced Exams in Geometry
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Do 5 out of the following 7 questions. Indicate clearly what questions you want to have graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1. Prove or disprove: a simply connected manifold is orientable.

Problem 2. Let $f : M \to \mathbb{R}$ be a smooth, proper function on a connected surface M. (Recall that f is *proper* iff the preimage of any compact set under f is compact.)

(1) What is the preimage of a regular value of f? (Hint: Use the classification of 1-manifolds.)
(2) Show that if f has no critical points, then M is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.

Problem 3. Consider the vector fields $X = y\partial_z - z\partial_y, \quad Y = z\partial_x - x\partial_z, \quad Z = x\partial_y - y\partial_x$ on $\mathbb{R}^3$.

(1) Describe the integral curves of X, Y and Z, as well as the corresponding flows on $\mathbb{R}^3$.
(2) Show that the span of X, Y and Z defines a rank 2 subbundle $E \subset TN$, where $N = \mathbb{R}^3 \setminus \{0\}$, and find a 1-form α on N whose kernel is E.
(3) Prove that N is foliated by leaves (integral manifolds) tangent to E; please compute these leaves explicitly.

Problem 4. Suppose L is a real line bundle over a compact manifold M. Which of the following vector bundles over M necessarily has a global non-vanishing section (proof or counterexample):

(1) The bundle L itself?
(2) The tensor product $L \otimes L$?
(3) The direct sum $L \oplus L$?
(4) The rank n bundle $L \oplus \cdots \oplus L$ for $n \geq 3$?

Problem 5. Consider the infinite dimensional vector space $\Omega^2(T^4)$ of all smooth 2-forms on the flat 4-torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$.

(1) Verify that the Hodge operator $*$ on $\Omega^2(T^4)$ satisfies $*^2 = 1$, and thus there is a decomposition $\Omega^2(T^4) = \Omega_+ \oplus \Omega_-$ into ± 1-eigenspaces, the *self-dual* and *anti-self-dual* 2-forms on T^4.
(2) Show that the harmonic 2-forms on T^4 comprise a vector subspace $H \subset \Omega^2(T^4)$ isomorphic to the constant-coefficient 2-forms on $\mathbb{R}^4$. (Recall that a form ω is *harmonic* iff $d\omega = 0 = d * \omega$.)
(3) Find a basis for H consisting of self-dual and anti-self-dual 2-forms.

Problem 6. The graph $z = f(x, y)$ of the function $f(x, y) = xy$ defines a smooth surface $\Sigma \subset \mathbb{R}^3$ in Euclidean space.

(1) Determine the induced Riemannian metric on Σ and show that it is complete.
(2) Compute the Gauss and mean curvatures of Σ at the origin.
(3) Describe parallel translation in Σ along the curve $\{x = 0\}$.

Problem 7. Let $G \subset GL(2, \mathbb{R})$ be the set of all 2-by-2 matrices A such that $A^tQA = Q$, where Q is the diagonal matrix with entries 1 and -1.

(1) Show that G is a Lie group, determine its Lie algebra and calculate its dimension.
(2) How many components does G have?
(3) Give an explicit parametrization of the identity component of G via the exponential map.