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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
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Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

PROBLEM 1

The equation for a vibrating string with interval damping is

$$u_{tt} = u_{xx} + \in u_{txx} \quad , \quad for \;\; 0 < x < 1, \; t > 0,$$
$$u(0,t) = u(1,t) = 0,$$

where $\in > 0$ is constant.

(a) Show that the energy

$$E(t) = \frac{1}{2}\int_0^1 (u_t^2 + u_x^2)dx$$

is decreasing in time.

(b) Show that there is at most one classical solution to the IBVP with

$$u(x,0) = u_0(x).$$

PROBLEM 2

Let $b : \mathbb{R}^r \times \mathbb{R}^\ell \to \mathbb{R}^r$ a bounded, continuous function satisfying

$$|b(x_1, y) - b(x_2, y)| \leq K|x_1 - x_2| \;\; for \; all \;\; x_1, x_2 \in \mathbb{R}^r$$

and K is a constant independent of y. Let $\xi : [0, \infty) \to \mathbb{R}^\ell$ a bounded and continuous function such that the following limit exists, uniformly in x:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T b(x, \xi(t))dt := \bar{b}(x)$$

Show that

(a) $\bar{b}$ is a Lipschitz function with Lipschitz constant K.

(b) If $X^\varepsilon = X^\varepsilon(t)$ solves

$$X^{\varepsilon\prime}(t) = b\left(X^\varepsilon(t), \xi\left(\frac{t}{\varepsilon}\right)\right) \quad , \quad X^\varepsilon(0) = x$$

and $\overline{x} = \overline{x}(t)$ solves

$$\overline{x'}(t) = \overline{b}(\overline{x}(t)) \quad , \quad \overline{x}(0) = x$$

Then

$$\max_{t=[0,T']} |X^\varepsilon(t) - \overline{x}(t)| \to 0 \quad as \;\; \varepsilon \to 0$$

on any finite interval $I = [0, T]$

PROBLEM 3

(a) Let $u(x,t)$ be a smooth solution of $u_{tt} = c^2 \Delta u$ in three dimensions, $x = (x_1, x_2, x_3)$, with initial data $u(x,0) = f(x), u_t(x,0) = g(x)$. For each fixed x, let

$$I(r,t) = \frac{1}{4\pi} \int_{|\xi|=1} u(x + r\xi, t) \, d\xi$$

be the spherical mean of u over a sphere of radius r centered at x. Show that for each fixed x, $I(r,t)$ satisfies the radially symmetric wave equation in three dimensions, $I_{tt} = c^2 \frac{1}{r^2}(r^2 I_r)_r$.

(b) Find the differential equation satisfied by $J(r,t) = rI(r,t)$, and use this equation to represent the solution $u(x,t)$ in terms of the initial data $f(x)$ and $g(x)$.

PROBLEM 4

(a) Suppose that $f(u,v)$ and $g(u,v)$ are smooth functions of two real variables, and that $f_v(u,v) < 0$ and $g_u(u,v) < 0$ for all (u,v). Show that if $(u_k(t), v_k(t))$, $k = 1,2$ are two solutions of the initial value problem

$$\begin{aligned} u' &= f(u,v) \\ v' &= g(u,v), \end{aligned} \tag{1}$$

and that if $w(t) = u_1(t) - u_2(t)$ and $z(t) = v_2(t) - v_1(t)$ are both positive (resp. both negative) at time $t = 0$, then $w(t)$ and $z(t)$ are both positive (resp. both negative) for all times $t \geq 0$.

(b) Show that (1) cannot have a nonconstant, periodic solution. (**Hint**: assume to the contrary that $(u_1(t), v_1(t))$ is such a solution with *minimal* period $T > 0$. Assume that the solutions parametrized so that $u_1(0)$ is the maximum of $u_1(t)$ and that $(u_2(t), v_2(t)) = (u_1(t+\tau), v_1(t+\tau))$ where $u_1(\tau)$ is the minimum value of $u_1(t)$ on $0 \leq t \leq T$.)

PROBLEM 5

Consider the equation

$$\begin{cases} u_t = u_{xx} - u & x \in (0,1) \quad , \quad t > 0 \\ u(x,0) = f(x) & x \in (0,1) \\ u(0,t) = u(1,t) = 0 \quad , \quad t \geq 0 \end{cases}$$

(a) Prove that if u is a smooth solution of (1) then the maximum and minimum values of u for $0 \le x \le 1$, $0 \le t \le T < \infty$ are attained either at $\{t = 0\}$ $\{x = 0\}$, or $\{x = 1\}$.

(b) Study the asymptotic behavior of the (smooth) solution u as $t \to \infty$.

PROBLEM 6

Consider the Sobolev space $H^S(\mathbb{R}^n)$, where $s \in \mathbb{R}$ and the space $S(\mathbb{R}^n)$ of rapidly decaying smooth functions.

(a) Prove that if $2S > n$ then there is a constant C depending only on such that

$$|u(x)| \le C\|u\|_s, \quad x \in \mathbb{R}^n$$

for all $u \in S(\mathbb{R}^n)$.

(b) Using (a) prove that any function $u \in H^s(\mathbb{R}^n)$ can be identified almost everywhere (with respect to the Lebesgue measure) with a bounded continuous function.