DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
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Work all problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (**20** pts) Give a precise **definition** for the following:
   (a) A complete family of density functions.
   (b) A Uniformly Most Powerful (UMP) test.
   (c) A regular exponential class of density functions.
   (d) State the Lehmann-Scheffe Theorem.

2. (**25** pts) Consider a simple random sample of size n from a Poisson distribution with mean μ. Let $\theta = P(X = 0)$.
   (a) Find the MLE of θ and show that it is a consistent estimator.
   (b) Let $T = \sum X_i$. Show that $\tilde{\theta} = [(n-1)/n]^T$ is an unbiased estimator of θ.
   (c) Find the UMVU estimator of θ.
   (d) Does the UMVU in (c) attain the CRLB for the variances of unbiased estimators of θ? Show work.

3. (**15** pts) The p.d.f. of an exponential distribution with mean θ is

$$f(x) = \theta^{-1} \exp(-x/\theta) \quad \text{for } x > 0, \text{ and 0 elsewhere.}$$

   Let $X_1, \ldots X_n$ be a random sample from this p.d.f.
   (a) Derive the MLE of θ. It is required to justify that your answer is indeed an MLE.
   (b) Give the MLE of θ^2, with justification (note that $\theta^2 = Var(X_i)$).
   (c) For a large n, find an approximate 95% confidence interval for θ and for θ^2, respectively.

4. (**20** pts) Suppose that $X_1, \ldots, X_n$ is a random sample from a $\mathrm{N}(\mu, \sigma^2)$ distribution, with μ and σ^2 unknown.

   (a) Write down, without proof, the MLEs of μ and σ^2, respectively.

   (b) Write down, without proof, the MLE of σ^2 given $\mu = \mu_0$.

   (c) Using the given sample, derive an α-level likelihood ratio test for $H_0: \ \mu = \mu_0$ against the alternative $H_1: \ \mu \neq \mu_0$, where μ_0 is a given number.

   (d) For a large n and $\alpha = 0.05$, find the asymptotic power of the test if $\mu = \mu_0 + 1$. You may use $\Phi(\cdot)$ to denote the c.d.f. of the $N(0,1)$ distribution.

5. (**20** pts) Let $X_1, \ldots, X_n$ be a random sample form an exponential distribution with density $f(x; \theta) = \theta e^{-\theta x}$, $x > 0$ (having mean $1/\theta$). Assume a prior density for θ which is also exponential with mean $1/\beta$, where β is known.

   (a) Prove that the posterior distribution of β is a Gamma distribution. *If you can't do part (a), assume the posterior distribution is Gamma with parameters a and b and do the remaining parts.*

   (b) Using squared error loss find the Bayes estimator of θ.

   (c) Using absolute error loss, find the Bayes estimator of θ (this won't have an explicit analytical expression but your answer can be expressed using a percentile of the gamma distribution.)

   (d) Derive a 95% Bayesian confidence interval for θ.

   (e) Derive a 95% Bayesian confidence interval for $\mu = 1/\theta$.