DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
FRIDAY - AUGUST 30, 2002

Do five of the seven problems. All problems carry equal weight. Passing level: 60% with at least two problems substantially correct.

1. Pat has \$50,000 in a bank account earning six percent interest (continuously compounded).
   (a) Write a differential equation and initial condition for the value, v, of her account at year t.
   (b) Solve the problem in part (a) and find the value of the account after twenty years.
   (c) Pat can instead, invest \$30,000 of her money in a tree farm whose value in twenty years is expected to be \$100,000. Her cost of insurance on the farm will be \$200 per year, withdrawn continuously in small amounts from her bank account. Write a new differential equation and initial condition for her account balance, if she makes this investment.
   (d) Solve the problem stated in part (c) and find the TOTAL value of her investment after 20 years.

2.

$$(IVP)\begin{cases} a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), & t > 0 \\ y(0) = y_0, \quad y^1(0) = y_1, \end{cases}$$

   (a, b, c, y_0, y_1 are specified constants, $f(t)$ is a given function)

   (a) Describe a physical problem modeled by this (IVP). Be clear about the meanings of each of the quantities $(a, b, c, y_0, y_1, y, t, f(t))$ for your example.
   (b) Solve (IVP) when $a = 1, b = 4, c = 5, f(t) \equiv 1, y_0 = 2$, and $y_1 = 0$.

3. Consider the circuit equation

$$\ddot{I} + 4\dot{I} + I = 0$$

   (a) Rewrite the equation as a 2-dimensional linear system.
   (b) Is the origin asymptotically stable?

4. For which values of the real parameter k does

$$(EVP)\begin{cases} (DE) & F'' + \lambda F = 0, \quad 0 < x < 1, \\ (BC) & F'(0) + kF(0) = 0, \quad F(1) = 0 \end{cases}$$

   have

   (a) a negative eigenvalue?
   (b) a zero eigenvalue?
   (c) a positive eigenvalue?

5. Solve Laplace's equation inside a square $0 \le x \le 1, \quad 0 \le y \le 1$ with the following boundary conditions;

$$\frac{\partial u}{\partial x}(0, y) = 0, \frac{\partial u}{\partial x}(1, y) = 0, \quad \frac{\partial u}{\partial y}(x, 1) = 0$$

$$u(x, 0) = \begin{cases} 0 & x > \frac{1}{2} \\ 1 & x < \frac{1}{2} \end{cases}$$

6.

$$(IBVP)\begin{cases} (DE) & u_t = u_{xx}, \quad 0 < x < \pi, t > 0 \\ (BC) & u(0, t) = 0, u_x(\pi, t) = 0 \\ (IC) & u(x, 0) = 7\sin(x/2) \end{cases}$$

   (a) Describe a physical problem modeled by this $(IBVP)$.
   (b) Solve this $(IBVP)$.

7. At time $t = 0$, an infinite string is plucked. It's initial displacement is $u(x, 0) = 0$ and its initial velocity is piecewise linear, given by

$$\begin{cases} 0 & , \quad x < -1 \\ -|x| + 1 & , \quad -1 \le x \le 1 \\ 0 & , \quad x > 1 \end{cases}$$

   a) Find the times t when $u(100, t)$ differs from 0.
   b) Find $\lim\limits_{t \to \infty} u(100, t)$