DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
TUESDAY, AUGUST 28, 2001 (3 Hours)

Note: *There are five problems, each 20 points. Sixty points are enough to pass at the Master's level and seventy-five at the Ph.D. level.*

1. Let $X_1, \dots X_n$ represent a collection of n random variables, with some distributions depending only on a parameter θ. Define precisely:

   (a) An exact $100(1-\alpha)\%$ confidence interval for θ.

   (b) A pivotal quantity for θ.

   (c) The mean squared error of an estimator $\hat{\theta}$ for θ.

   (d) What it means for a sequence of estimators $\hat{\theta}_n$ to converge in probability to θ? What other name do we give to $\hat{\theta}_n$ if it converges in probability to θ?

   (e) A $100(1-\alpha)\%$ Bayesian confidence interval for θ.

2. Let $X_1, \dots X_n$ be a random sample from a geometric distribution with density $f(x) = P(X_i = x) = \theta(1-\theta)^{x-1}$, for $x = 1, 2, \dots$, where $0 \leq \theta \leq 1$. The mean of this distribution is $1/\theta$ and the variance is $\sigma^2 = (1-\theta)/\theta^2$.

   (a) Find the maximum likelihood estimator of θ; denote it by $\hat{\theta}$. Be sure to verify that you have maximized the likelihood and discuss whether there is ever an issue with the existence of the MLE.

   (b) Find the Cramer-Rao lower bound for unbiased estimators of θ, and then give the approximate distribution of $\hat{\theta}$ as n gets large.

   (c) Use the result in part (b) to develop an approximate confidence interval for θ.

   (d) The population coefficient of variation is $\gamma = \sigma/\mu = (1-\theta)^{1/2}$. Find the MLE for γ, obtain their approximate large sample distribution of the MLE and develop an approximate confidence interval for γ.

3. Suppose that $X_1, \dots, X_n$ are random samples from a $\mathrm{N}(\mu, 1)$ distribution.

   (a) Derive the MLE of μ (needs to check that it is indeed an MLE), then obtain (by using known results) the MLE of μ^2.

   (b) Find the uniformly minimum variance unbiased estimator of μ^2. State carefully what result(s) you are using.

   (c) Derive a size α likelihood ratio test for the null hypothesis $H_0: \ \mu^2 = 0$ against the alternative hypothesis $H_1: \ \mu^2 > 0$. (Hint: convert this to a test for μ.)

   (d) Obtain the power function of the test in (c), with $\alpha = 0.05$, in terms of μ and $\Phi(\cdot)$ (the cdf of the $N(0,1)$ distribution).

4. Let $X_1, \ldots, X_n$ be random samples from an exponential distribution with pdf $f(x) = \mu^{-1}\exp(-x/\mu)$ over $x \geq 0$ (0 otherwise) with $\mu > 0$. In this case, μ is the mean.

   (a) Find an exact 95% confidence interval for the survival probability $\Pr[X_1 > t_0]$, using all observations, where $t_0 (> 0)$ is fixed and known. (Hint: make use of the sample mean, and the fact that $2X_1/\mu$ has a chi-square distribution with 2 degrees of freedom. You need not prove this result.)

   (b) Derive the most powerful test of size 0.05 for testing $H_0: \ \mu = \mu_0$ against $H_1: \ \mu = \mu_1$, where μ_0 and μ_1 are known constants, $0 < \mu_0 < \mu_1$. A complete answer should include an explicit description of the critical region, using a known distribution. Which theorem did you use to claim that the test is most powerful? Give the name of the theorem and state the theorem.

   (c) Explain why the test in part (b) is also a uniformly most powerful size 0.05 test for testing $H_0 : \mu \leq \mu_0$ against $H_1 : \mu > \mu_0$.

5. Suppose you have a random sample of size n from a Uniform($\theta, \ \theta+1$) distribution, $\theta \geq 0$. Let $X_{(1)}$ and $X_{(n)}$ denote the minimum and maximum sample value respectively. To test the null hypothesis $\theta = 0$ versus the one-sided alternative $\theta > 0$, use the test that rejects the null hypothesis if $X_{(1)} \geq c$ or $X_{(n)} \geq 1$. It is known that the joint pdf of the extremes $Y_1 = X_{(1)}$ and $Y_n = X_{(n)}$ is given by

$$g(y_1, y_n) = n(n-1)[F(y_n) - F(y_1)]^{n-2} f(y_1) f(y_n),$$

   for $0 \leq \theta < y_1 < y_n \leq \theta + 1$, and is zero otherwise, where $F(\cdot)$ and $f(\cdot)$ are, respectively, the cdf and pdf of the said uniform distribution.

   (a) Write the joint pdf $g(y_1, y_n)$ explicitly in terms of y_1, y_n, n and θ (i.e., no F and no f), and draw a diagram showing the domain where $g(y_1, y_n) \neq 0$.

   (b) Determine c such that the test will have size α.

   (c) Find the power function of the test.

   (d) Let $\alpha = .10$. Find the necessary sample size so that the test will have power at least .80 if $\theta \geq 1$.