DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - NUMERICS
August, 2001

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. Let A, B and C be matrices of orders $m \times n$, $n \times p$ and $p \times q$ respectively. Do an operations count for computing $A(BC)$ and $(AB)C$. Under what conditions is one computation preferable to the other?

2. The *false position method* is a variation on the bisection method, where the next test point c of the bracketing interval $[a, b]$ is the point where the line through the points $(a, f(a))$ and $(b, f(b))$ crosses the x-axis (rather than $c = \frac{a+b}{2}$ for bisection).

   (a) Write down the algorithm for a general function in detail.

   (b) Compare this method to the secant method. Give an example showing that this is different from the secant method (no need to give f , a picture with a few words should do).

3. Let f be an even function, so $f(-x) = f(x)$ for all x. Find the best interpolation of f on the interval $[-1, 1]$ using only the values $f(0)$ and $f(1)$. Also find a bound for the error.

4. We want to approximate

$$\int_0^1 f(x)\, x \, dx$$

by a rule of the form

$$a\, f(b).$$

Find a and b so that the method is exact for as many polynomials as possible. Also find the error term.

5. For the trapezoidal rule (denoted by I^T) for evaluating

$$I = \int_a^b f(x)\, dx,$$

we have the asymptotic error formula

$$I^T = I + C_1 h^2 + O(h^4),$$

and for the midpoint formula I^M, we have

$$I^M = I + C_2 h^2 + O(h^4),$$

where C_1 and C_2 are constants. Using these results, obtain a new numerical integration formula $\widetilde{I}$ combining I^T and I^M with a higher order of convergence. Write out the weights to the new formula $\widetilde{I}$.

6. (a) Write the $2nd$ order ordinary differential equation

$$y''(x) + 1001\, y'(x) + 1000\, y(x) = 0$$

as a first order system.

(b) Write down Euler's method to solve this system.

(c) For what step sizes h will this method be stable? Justify your answer.

7. Find the polynomial $q(x) = a_0 + a_1 x$ of degree one or less which approximates e^x best in the sense that it minimizes the error

$$\int_0^1 (e^x - q(x))^2 \, dx.$$