Department of Mathematics and Statistics
University of Massachusetts
Basic Exam - Complex Analysis
August 2001

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All closed curves are assumed to be oriented counterclockwise.

1. For all positive radii R other than 2 and 3, find the value of the integral

$$\int_{C_R} \frac{z^3+2}{(z-3)(z^2+4)}dz,$$

where C_R is the circle $|z| = R$.

2. Let U be the first quadrant of the open unit disk

$$U \;=\; \{z \;:\; |z| < 1,\; \mathrm{Re}(z) > 0,\; \mathrm{Im}(z) > 0\}.$$

Find a one-to-one conformal map from U onto itself, which extends continuously to the closure $\bar{U}$, and permutes the "edges" of ∂U cyclically and counterclockwise (so, the real-line-segment $[0, 1]$ is mapped onto the first quadrant of the unit circle, etc). Justify your answer!

Calculate

$$\int_0^\pi \frac{dx}{2+\cos^2(x)}$$

3. Evaluate the integral

$$\int_0^\infty \frac{x^{1/3}}{x^2+9x+8}dx.$$

Justify all your steps.

4. Show that the series

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[\frac{1}{z-n} + \frac{1}{n}\right]$$

defines a meromorphic function on the complex plane and determine the set of poles.

5. Find the number of solutions of the equation

$$30z^4 \quad = \quad (z^2 - 1)\cos(z)$$

in the unit disk $|z| < 1$.

6. Using the maximum modulus principle prove the Schwarz lemma:

   (a) If $f(z)$ is analytic on the unit disc $D = \{z \in \mathbb{C}, \ |z| < 1\}$ and satisfies $f(0) = 0$, and $|f(z)| < 1$ for $z \in D$; then $|f(z)| \leq |z|$ for any $z \in D$, and $|f'(0)| \leq 1$.

   (b) Moreover, if one also knows that $|f(a)| = |a|$ for some $0 \neq a \in D$, then: $f(z) = \lambda \cdot z$ for some complex number λ of modulus one.

7. Let U be a simply connected open subset of the complex plane. Show that for any two points p, q in U, there exists a one-to-one holomorphic map f from U onto itself such that $f(p) = q$.

8. Prove or disprove the following statements:

   (a) For any open subset W of the complex plane, any harmonic function on W is the real part of a holomorphic function on W.

   (b) If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are four distinct points on the Riemann sphere and $\beta_1, \beta_2, \beta_3, \beta_4$ are another four distinct points on the Riemann sphere, there is a fractional linear transform F such that $F(\alpha_i) = \beta_i, \ i = 1, 2, 3, 4$.

9. Find a Laurent series expansion of

$$\frac{z^2 + 2z + 5}{(z+2)(z^2 + 2z + 1)}$$

in an annulus about $z = -1$, which contains -3. What is the largest such annulus?