DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - MATHEMATICAL STATISTICS
FRIDAY, AUGUST 31, 2001

Do all six problems. All problems are equally weighted. A total of 70 points is enough to pass.

1. Let $X_n = 1$ with probability p_n, and $X = 0$ with probability $q_n = 1 - p_n$.

   (a) Suppose the random variables $X_1, X_2, \ldots$ are independent. Show that $\sum p_n < \infty$ is a necessary and sufficient condition for $\sum X_n < \infty$ a.s.

   (b) If the random variables are not assumed to be independent, is the condition in part (a) still sufficient?

   (c) Let Y be a nonnegative (finite) integer-valued random variable and define $X_n = I_{[Y \geq n]}$, where I denotes an indicator function. Show that, if $E(Y) = \infty$, the rvs X_n provide an example for which $\sum X_n < \infty$ a.s., but $\sum p_n = \infty$; thus the condition in part (a) is not necessary in this case.

   (d) Give an example of a random variable Y satisfying the description in part (c).

2. (a) Let X, Y, Y_n, $n \geq 1$, be random variables such that, for each n, Y_n and X are independent, and $Y_n \xrightarrow{D} Y$ (this denotes convergence in law, i.e., in distribution). Show that $Y_n + X \xrightarrow{D} Y + X$, where Y is independent of X.

   (b) Suppose X_n has the same distribution as X for each $n \geq 1$, and that X_n and Y_n are independent for each n. Is it true that $Y_n + X_n \xrightarrow{D} Y + X$?

   (c) Construct an example that shows that the result in part (a) can fail when Y_n and X are not independent. (Hint: Take X to be a symmetric rv, e.g., $N(0, 1)$.)

   (d) Let $\bar{X}$ and S^2 be the sample mean and sample variance of n i.i.d. $N(\mu, \sigma^2)$ rvs, with $\sigma^2 > 0$. It can be shown that $\sqrt{n}(S - \sigma) \xrightarrow{D} N(0, \sigma^2/2)$. Use this to find the limiting distribution of $\sqrt{n}(\bar{X} - \mu + S - \sigma)$. Justify your answer.

3. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. $\sim Poisson(\lambda)$, where $\lambda > 0$.

   (a) Find the maximum likelihood estimator $\hat{\theta}$ of $\theta = 1/\lambda$ based on $Y_1, Y_2, \ldots, Y_n$.

   (b) Determine the variance of $\hat{\theta}$.

   (c) Determine the asymptotic distribution of $\hat{\theta}$. (Hint: What is the asymptotic distribution of $\hat{\lambda}$?)

   (d) Let $\sigma^2(\theta)$ denote the variance of the asymptotic distribution of $\hat{\theta}$. Determine whether the following statement is true or false: $\sigma^2(\theta) = \lim_{n\to\infty} Var(\hat{\theta})$.

4. (a) State the Rao-Blackwell Theorem regarding the UMVU estimator.

   (b) Show that the sample variance based on i.i.d. r.v.'s is unbiased for the variance of any distribution as long as the variance is finite.

   (c) Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s from Binomial$(1, \theta)$.

      i. Do your best to justify why $T = \sum_{i=1}^{n} X_i$ is a complete, sufficient statistic for θ.

      ii. Find a UMVU estimator for the variance of X_1. Justify your answer. (Hint: You may need to use a property of a 0-1 variable.)

5. Let $X_1, \cdots, X_n$ denote random samples from an one-parameter exponential distribution with unknown mean μ.

   (a) Derive the most powerful test of size 0.05 for testing $H_0: \ \mu = \mu_0$ against $H_1: \ \mu = \mu_1$, where μ_0 and μ_1 are known constants, $0 < \mu_0 < \mu_1$. A complete answer should include an explicit description of the critical region, using a known distribution. [Hint: $(2/\mu)\sum_{i=1}^{n} X_i$ has a chi-square distribution with $2n$ degrees of freedom. You need not prove this result.]

   (b) Which theorem or lemma did you use to claim that the test in (a) is most powerful? Give the name and the content of the theorem.

   (c) Argue that the test obtained in (a) is also a uniformly most powerful size 0.05 test for testing $H_0 : \mu \leq \mu_0$ against $H_1 : \mu > \mu_0$.

   (d) Express the power of the test in (c) in terms of the CDF, say $G(\cdot)$, of a chi-square r.v. with $2n$ degrees of freedom. Is the power a decreasing or increasing function of μ? Explain.

6. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, where all parameters are unknown. Let S_1^2 and S_2^2 denote the sample variances of the two samples, respectively. Derive the α-level likelihood ratio test for $H_0 : \sigma_1^2 = \sigma_2^2$ *against* $H_1 : \sigma_1^2 \neq \sigma_2^2$. The resulting test should be in terms of a well-known statistic. (Note: Derivations of MLEs and well-known distributions are not required, and at the last stage of the derivation, a graph is helpful in determining the rejection region.)