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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - TOPOLOGY
1 SEPTEMBER 2000

Answer five of seven questions. Indicate clearly which five questions you want to have graded. Justify your answers.

Passing Standard:
For Master's level, 60 percent with two questions essentially complete.
For Ph.D. level, 75 percent with three questions essentially complete.

1. Consider the 1-dimensional compact, connected subspaces (graphs) R, O and B of $\mathbb{R}^2$ suggested by the corresponding letters of the alphabet (the space O is, of course, homeomorphic to the circle S^1). Decide whether or not any of these spaces are homeomorphic to each other.

2. Let X and Y be compact, connected spaces. Prove that $X \times Y$ is compact and connected.

3. Let X be any space, and let Y be a compact Hausdorff space. Suppose $f : X \to Y$ is a *proper* map (that is, for each compact $K \subset Y$ its preimage $f^{-1}(K) \subset Y$ is also compact). Show that
(i) $f(X)$ is closed in Y;
(ii) if f is injective, it is an embedding;
(iii) if f is bijective, it is a homeomorphism.

4. Suppose A is a closed subset of a complete metric space M. Show that A is complete in the induced metric.

5. Prove that a compact metric space is second countable.

6. Consider the sequence of functions $f(x) = x^n : \mathbb{R} \to \mathbb{R}$. On what subsets of $\mathbb{R}$ does this sequence converge in the topology of pointwise convergence? in the compact-open topology?

7. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that if X is compact, then any continuous function $f : X \to Y$ is also uniformly continuous, i.e., for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $a, b \in X$ one has $d_X(a, b) < \delta \Rightarrow d_Y(f(a), f(b)) < \varepsilon$.