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UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - DIFFERENTIAL EQUATIONS
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Do five of the following problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1. a) Let A be a real, symmetric $n \times n$ matrix with negative eigenvalues $\lambda_1, \ldots \lambda_n, \lambda_i < -p < 0, i = 1, \ldots, n$ for some $p > 0$. Prove that every solution $u(t)$ of $u' = Au$ satisfies

$$|u(t)| \leq He^{-pt}|u(0)|$$

for some H depending only on A; here $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. (Prove the result: do not simply cite a theorem).

b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth and satisfies $|f(u)| \leq k|u|^2$ as $|u| \to 0$. Construct a Liapunov function $V(u)$ for solutions of

$$u' = Au + f(u),$$

where A is as in a), on some small neighborhood of $u = 0$. Prove that this function strictly decreases on nonconstant solutions.

2. Consider the family of solutions $u_\epsilon(x)$ to the (infinite-domain) boundary-value problem:

$$\begin{cases} -\frac{d^2 u_\epsilon}{dx^2} + c^2 u_\epsilon = f_\epsilon(x) & (\epsilon > 0) \\ \lim_{x \to \pm\infty} u_\epsilon(x) = 0, \end{cases}$$

where $f_\epsilon(x) = \frac{1}{\epsilon} F\left(\frac{x}{\epsilon}\right)$ and $F(x)$ satisfies (i) $F(x) \geq 0$, (ii) $F(x) = 0$ for $|x| \geq 1$, and (iii) $\int_{-1}^{+1} F(x)dx = 1$.

a) Determine the limit solution $u_*(x) = \lim_{\epsilon \to 0+} u_\epsilon(x)$

b) What equation does the function

$$v(x) = \int_{-\infty}^{+\infty} u_*(x-y)\phi(y)dy$$

satisfy, given an arbitrary continuous and integrable function ϕ?

3. Let $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a flow on $\mathbb{R}^n$, so that $\phi_t(x)$ is the trajectory at time t with initial data x.

   a) Prove that if the ω-limit set $\omega(\phi_t(x))$ of a trajectory $\phi_t(x)$ lies in a bounded subset of $\mathbb{R}^n$ then $\omega(\phi_t(x))$ is closed.

   b) Prove that if $p \in \omega(\phi_t(x))$ then $\phi_t(p) \in \omega(\phi_t(x))$ for each t.

4. Consider the PDE for a vibrating string with a particular damping term:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \mu \frac{\partial^3 u}{\partial x^2 \partial t} & (0 < x < 1) \\ u(0,t) = 0 = u(1,t), & \end{cases}$$

   with a positive coefficient μ. Show that the initial-boundary value problem for this equation has a unique solution. **Hint:** Find an energy function E and verify that $\frac{dE}{dt} \leq 0$].

5. a) Demonstrate that the solution to the variational problem

$$min \int_\Omega | \bigtriangledown u|^2 dx \text{ subject to } \int_\Omega u^2 dx = 1, u \in H_0^1(\Omega)$$

   coincides with the first eigenfunction $u = \varphi_1$ for $-\Delta$ on Ω; namely,

$$\begin{array}{ll} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega \subset R^n, \\ \varphi_1 = 0 & \text{on } \partial\Omega, \end{array}$$

   and that the minimum is attained when $u = \varphi_1$, where φ_1 is the principal eigenfunction. Assume Ω is a bounded domain with smooth boundary.

   b) Use the characterization in (a) to show that if the domain Ω is a strict subdomain of another domain $\tilde{\Omega}$, then

$$\lambda_1(\Omega) \geq \lambda_1(\tilde{\Omega}),$$

   where these are the first eigenvalues for each domain.

6. Let $(x(t), y(t))$ be a solution of the system

$$x' = x((x-1)(2-x) - y)$$
$$y' = y(-d - (y-c)^2 + x)$$

with positive initial data $x(0) > 0,\ y(0) > 0$. Here c, d are positive constants.

a) Prove that the solution remains uniformly bounded, $0 \le x(t) \le M, 0 \le y(t) \le N$ for all $t \ge 0$ and some $M, N > 0$ (depending on the data).

b) Suppose that the equation

$$x = d + [(x-1)(2-x) - c]^2$$

has NO real roots. Show that every solution $(x(t), y(t))$ with positive initial data tends to a rest point on the nonnegative x-axis as $t \to +\infty$.

7. a) State and prove the classical maximum principle for the parabolic initial-boundary value problem

$$\begin{cases} u_t = \Delta u + au & (x,t) \in (0,\pi) \times x(0,T) \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

when $a < 0$.

b) When $a > 1$, give a counterexample to the statement in part (a).