Basic Exam – Complex Analysis
August 31, 2009

Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

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1. (a) Find all values of i^i and indicate which is the principal value.

   (b) Locate the poles of $\dfrac{\sinh z}{z^5}$ and determine the order of each.

2. (a) Let f be a non-constant holomorphic function on the closed unit disk D. Suppose $|f(z)| = e$ for all z on the unit circle. Prove that f has a zero at some point of D.

   (b) Suppose that f is holomorphic inside an on the unit circle $|z| = 1$ and satisfies $|f(z)| < 1$ for $|z| < 1$. Show that the equation $f(z) = z^3$ has exactly three solutions (counting multiplicities) inside the unit circle.

3. (a) Determine the image of the annulus

$$A = \{ z : 1 \le |z| \le e \}$$

   under the principal logarithm function Log. Fully justify your answer.

   (b) Obtain an explicit formula for a conformal map of the open set

$$S = \{ z \in \mathbf{C} : 0 < \operatorname{Arg} z < \frac{\pi}{2},\ 0 < |z| < 1 \}$$

   onto the open upper half plane; note that Arg denotes the principal branch of arg.

4. By integrating a suitable complex-valued function around a suitable contour, find the (principal) value of the real integral:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 4}\, dx$$

5. Find the Laurent expansion of

$$f(z) = \frac{1}{z^4 + 13z^2 + 36}$$

   in an annulus (centered at the origin) that contains $\sqrt{5}$. What is the largest possible annulus on which the expansion is valid?

6. Suppose $a > 1$ is a real number. Prove via the Residue Theorem that

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2} d\theta = \frac{2\pi a}{(a^2 - 1)^{3/2}}.$$

7. Suppose f is entire and maps every unbounded sequence to an unbounded sequence. Show that f is a polynomial.

8. Let f be a holomorphic map of the open unit disc D into itself which is not the identity map $f(z) = z$. Show that f can have at most one fixed point in D.

9. Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n > 0$. Prove that

$$\frac{1}{2\pi i} \int_C z^{n-1} |f(z)|^2 dz = a_0 \overline{a_n} R^{2n},$$

where C is the circle $|z| = R$ traversed once counterclockwise.

10. Let $f\colon \mathbf{C} \to \mathbf{C}$ be a meromorphic function that has periods 1 and τ with $\mathrm{Im}(\tau) > 0$. Thus $f(z + j + k\tau) = f(z)$ for all $z \in C$ and all $j, k \in \mathbf{Z}$.

    (a) Prove that if f has no singularities whatsoever, then f must be constant.

    (b) Assume now that f is not constant. Let m be the total number of poles of f in the set
    $$S = \{\, s + t\tau \ : 0 \le s < 1,\ 0 \le t < 1\},$$
    counting multiplicities What is the total number of zeros of f in S, again counting multiplicities, and why?