DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
TUESDAY, JANUARY 20, 2004

*Work all five problems, each worth 20 points. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.*

1. Let $X_1, \cdots, X_n$ be a random sample from a Pareto distribution with p.d.f.

$$f(x, \theta) = 2\theta^2/x^3, \quad x \geq \theta; \quad 0, \quad \text{otherwise}.$$

Here, $\theta$ ($> 0$) is a parameter.

(a) Find the MLE $\hat{\theta}$ of $\theta$. What is the MLE of $\theta^2$?

(b) Define consistency of an estimator in general.

(c) Find the method of moments estimator of $\theta$. Is this estimator consistent? (You may use the weak law of large numbers.)

2. Let $X_1, \cdots, X_n$ be a random sample from a Bernoulli distribution with p.m.f.

$$f(x, \theta) = \theta^x (1 - \theta)^{1-x}, \quad x = 0, 1; \quad 0, \quad \text{otherwise}.$$

Here, $\theta$ ($0 \leq \theta \leq 1$) is a parameter.

(a) Show that the sample variance $S^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$ can be written as

$$S^2 = \frac{T(\vec{X})}{n(n-1)} [n - T(\vec{X})],$$

where $T(\vec{X}) = \sum_{i=1}^{n} X_i$.

(b) Let $\sigma^2(\theta) = \theta(1 - \theta)$, the common variance of the $X_i$'s. Find the UMVUE of $\sigma^2(\theta)$ using (a), or otherwise. State carefully all the results you use.

3. Let $X_1, \cdots, X_{100}$ be i.i.d. Poisson($\lambda$) random variables, where $\lambda$ ($> 0$) is the mean as well as the variance of $X_1$. Let $\overline{X}$ denote the sample mean.

(a) Applying the central limit theorem to $\overline{X}$, find an approximate pivotal quantity for $\lambda$.

(b) Derive an approximate 95% confidence interval for $\lambda$ in terms of $\overline{X}$.

4. Let $X_1, \cdots, X_n$ be a random sample from a normal distribution with mean $\mu$ and standard deviation $\sigma$, where both $\mu$ and $\sigma$ are unknown. Derive the likelihood ratio test for testing

$$H_0 : \ \mu = \mu_0 \ \text{ against } \ H_1 : \ \mu \neq \mu_0,$$

where $\mu_0$ is a given number. Specify the critical value and critical region for the test of size $\alpha$.

5. Let $X_1, \cdots, X_{25}$ be a random sample from a normal distribution with an unknown mean $\mu$ and variance 1. Consider testing the hypotheses

$$H_0 : \ \mu \leq 0 \ \text{ against } \ H_1 : \ \mu > 0.$$

It is known that the UMP size 0.05 test rejects $H_0$ iff $5\bar{X} > 1.645$.

(a) Explain what it means for the test to have size 0.05, and what it means to be UMP.

(b) Construct the power function of the test, and calculate the power of the test at $\mu = 0.5$. Is the power function an increasing function of $\mu$? (Explain.)

(c) What is the Type I error probability of the test at $\mu = 0$? Is this probability larger or smaller than the Type I error probability at $\mu = -1$? (Explain briefly.)

(d) Now suppose we change the critical value 1.645 in the test to 1.96. Compute the size of the new test.